UNIT 6

Patterns in Shape

Shape is an important and fascinating aspect of the world in which you live. You see shapes in nature, in art and design, in architecture and mechanical devices. Some shapes, like the Rock and Roll Hall of Fame building, are three-dimensional. Others, like the architect's plans for the building are two-dimensional.

In this unit, your study will focus on describing and classifying two-dimensional and three-dimensional shapes, on visualizing and representing them with drawings, and on analyzing and applying their properties. You will develop understanding and skill in use of the geometry of shape through work on problems in three lessons.

Lessons

1 *Two-Dimensional Shapes*

Use combinations of side lengths and angle measures to create congruent triangles and quadrilaterals. Investigate properties of these figures by experimentation and by careful reasoning. Use those properties to study the design of structures and mechanisms and to solve problems.

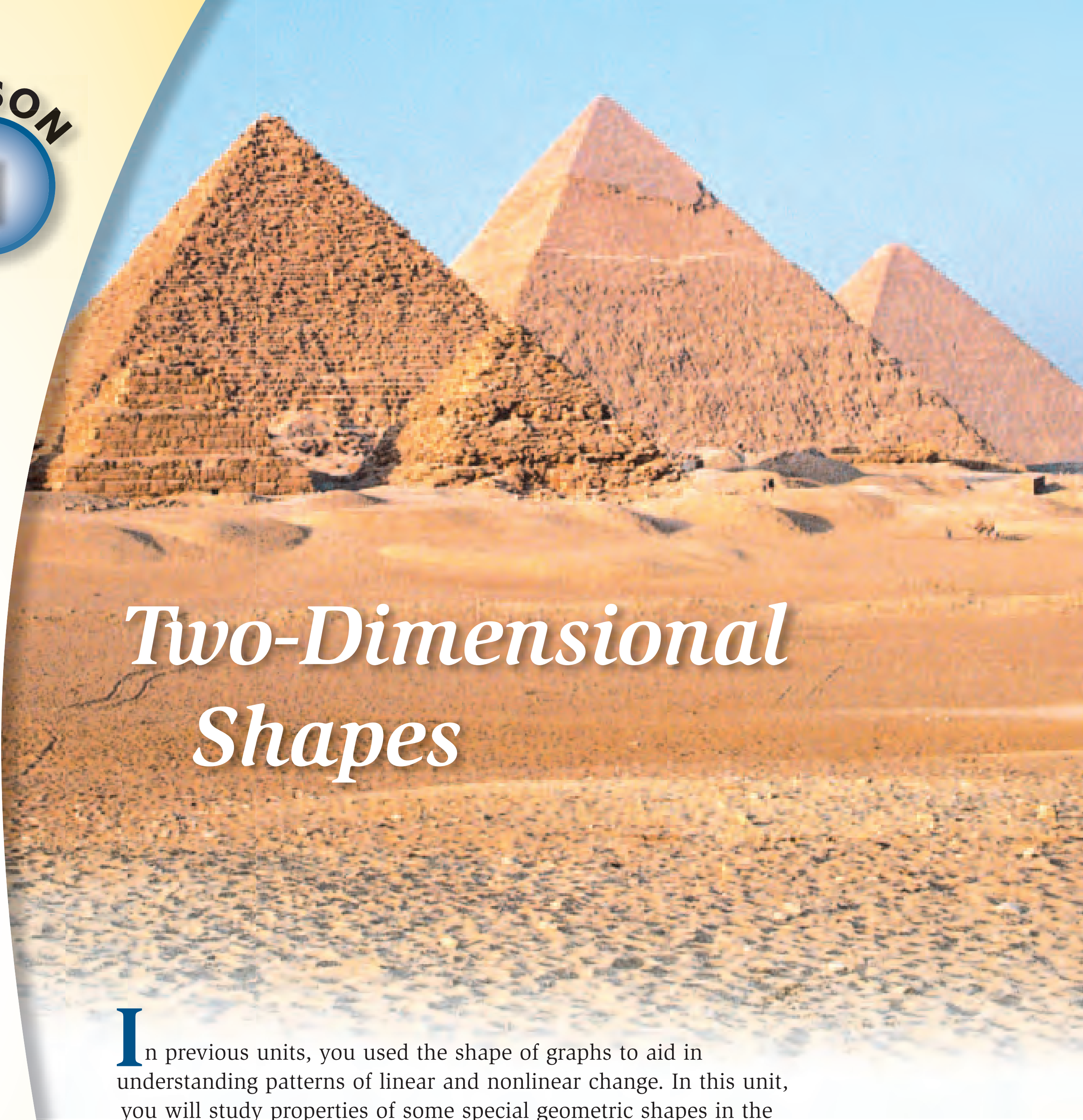
2 *Polygons and Their Properties*

Recognize and use symmetry and other properties of polygons and of combinations of polygons that tile a plane.

3 *Three-Dimensional Shapes*

Recognize, visualize, and develop drawing methods for representing three-dimensional shapes. Analyze and apply properties of polyhedra.

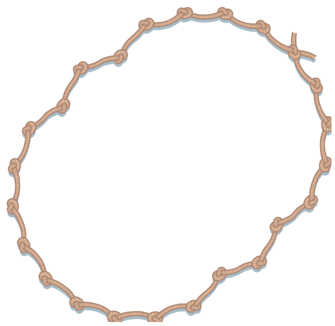
LESSON 1



Two-Dimensional Shapes

In previous units, you used the shape of graphs to aid in understanding patterns of linear and nonlinear change. In this unit, you will study properties of some special geometric shapes in the plane and in space. In particular, you will study some of the geometry of two-dimensional figures called *polygons* and three-dimensional figures called *polyhedra*, formed by them.

The geometry of shape is among some of the earliest mathematics. It was used in ancient Egypt to construct the pyramids and to measure land. For example, when the yearly floods of the Nile River receded, the river often followed a different path. As a result, the shape and size of fields along the river changed from year to year. It is believed that the Egyptians used ropes tied with equally-spaced knots to re-establish land boundaries. To see how a knotted rope might be used in building design and measuring, think about how you could use a piece of rope tied into a 24-meter loop with knots at one-meter intervals.



CORBIS

Think About This Situation

Suppose that you and two or three friends each grabbed the rope at a different knot and pulled outward until the loop formed a particular shape.

a How could you position yourselves so that the resulting shape was an equilateral triangle? An isosceles triangle? A right triangle?

b How are the perimeters of the three triangles related? How do you think the areas are related?

c How could you position yourselves so that the resulting shape was a square? A rectangle? A parallelogram that is not a rectangle?

d How are the perimeters of the three quadrilaterals related? How do you think the areas are related?

As you complete the investigations in the following three lessons, you will discover why some shapes are used so frequently in building and in design. You will also discover how knowledge of a few basic properties of geometric figures can be used to reason to many additional properties of those shapes.

Investigation 1 Shape and Function

Buildings and bridges, like most objects around you, are three-dimensional. They have length, height, and depth (or width). To better understand the design of these objects, it is often helpful to examine the two-dimensional shapes of their components. Triangles and special quadrilaterals such as rectangles are among the most commonly occurring two-dimensional shapes in structural designs.

Brand X Pictures/AGE Fotostock

In this first investigation, you will explore conditions on the sides of triangles and quadrilaterals that affect their shape. In the process, you will discover some physical properties of these shapes that have important applications. As you work on the following problems, look for answers to these questions:

What conditions on side lengths are needed to build triangles and quadrilaterals? What additional constraints are needed to build special quadrilaterals?

Why and how are triangles used in the design of structures like bridge trusses?

Why and how are quadrilaterals used in the design of devices like windshield wipers?

1 Using strands of uncooked thin spaghetti, conduct the following experiment at least three times. Keep a record of your findings, including sketches of the shapes you make.

- Mark any two points along the length of a strand of spaghetti and break the spaghetti at those two points.
- Try to build a triangle with the pieces end-to-end.
- If a triangle can be built, try to build a differently shaped triangle with the same side lengths.

a. Was it possible to build a triangle in each case? If a triangle could be built, could you build a differently shaped triangle using the same three segments? Compare your findings with those of others.

b. If a triangle can be built from three segments, how do the segment lengths appear to be related? Use a ruler and compass to test your conjecture for segments of length 3 cm, 4 cm, and 5 cm. For segments of length 5 cm, 6 cm, and 12 cm. Revise your conjecture if needed.

c. Suppose a, b, and c are side lengths of *any* triangle. Write an equation or inequality relating a, b, and c. How many different equations or inequalities can you write relating a, b, and c?

d. Write in words the relationship that must be satisfied by the side lengths of any triangle (do not use letters to name the side lengths). This relationship is called the **Triangle Inequality**.

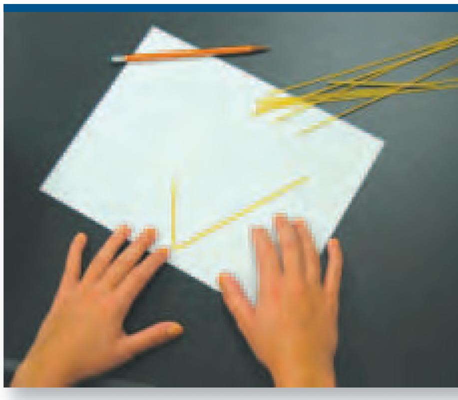
2 You may recall from your prior mathematical study that triangles can be classified in terms of their sides as *scalene*, *isosceles*, or *equilateral*.

a. What type of triangle were most of the triangles that you built in your experiment? Explain as carefully as you can why you might expect that result.

b. Draw an isosceles triangle that is not equilateral. Suppose a is the length of two of the sides and b is the length of the *base* of your triangle. How must a and b be related? Explain your reasoning using the Triangle Inequality.

c. Can you build an equilateral triangle of any side length a? Explain your reasoning using the Triangle Inequality.

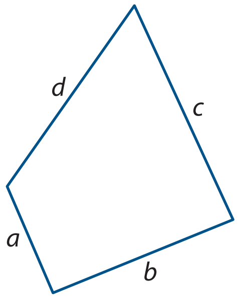
3 Now use strands of spaghetti to conduct this quadrilateral-building experiment at least three times. Keep a record of your findings, including sketches of the shapes you make.



- Mark any three points along the length of a strand of spaghetti and break the spaghetti at those three points.
- Try to build a quadrilateral with the pieces end-to-end.
- If a quadrilateral can be built, try to build another, differently shaped quadrilateral with the same side lengths.

a. Was it possible to build a quadrilateral in each case? If a quadrilateral could be built, could you build a differently shaped quadrilateral using the same four segments? Compare your findings with those of others.

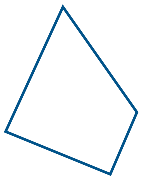
b. If a quadrilateral can be built from four side lengths, how are the side lengths related? Use a ruler and compass to test your conjecture for segments of length 3 cm, 5 cm, 8 cm, and 10 cm. For segments of length 4 cm, 4 cm, 7 cm, and 15 cm. For segments of length 2 cm, 4 cm, 8 cm, and 16 cm.



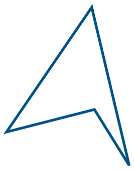
c. Suppose *a, b, c*, and *d* are consecutive side lengths of any quadrilateral. Write an equation or inequality relating *a, b, c*, and *d*. How many different equations or inequalities can you write relating *a, b, c*, and *d*?

d. Write in words the relationship that must be satisfied by the four side lengths of any quadrilateral (do not use letters to name side lengths).

Quadrilaterals are more complicated than triangles. They have more sides and more angles. In Problem 3, you discovered that using the same four side lengths of a quadrilateral, you could build quite different shapes. Quadrilaterals are classified as *convex*—as in the case of the quadrilateral below on the left—or *nonconvex*—as in the case of the quadrilateral on the right.



Convex



Nonconvex

4 Some special convex quadrilaterals can be characterized in terms of side lengths. For example, in completing Part c of the Think About This Situation, you likely created a **parallelogram** by forming a quadrilateral with opposite sides the same length.

a. Show how you can build a parallelogram using four segments cut from a strand of spaghetti and placed end-to-end.

  i. How many differently shaped parallelograms can you build with those four segments?

  ii. What additional constraint(s) would you have to build into the shape for it to be a rectangle?

Jim Laser

b. A **kite** is a quadrilateral with two distinct pairs of consecutive sides the same length.

i. Build a kite using the same four segments of spaghetti, in Part a, placed end-to-end.

ii. How many differently shaped kites can you build with those four pieces?

c. A **rhombus** is a quadrilateral with all four sides the same length.

i. Build a rhombus using four segments from a strand of spaghetti placed end-to-end.

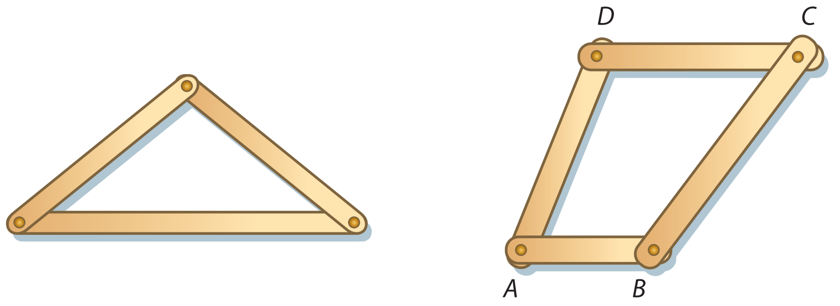
ii. How many differently shaped rhombi can you build with those four pieces?

iii. Explain why a rhombus is a parallelogram.

iv. What additional constraint(s) would you have to build into a rhombus for it to be a square?

The results of your experiments in building triangles and quadrilaterals lead to important physical applications.

5 Working with a partner, use plastic or cardboard strips and paper fasteners to make each of the models shown below.



a. Can you change the shape of the triangle model? Can you change any of the features of the model? Explain.

b. What features of the quadrilateral model can you change? What features of the model cannot change?

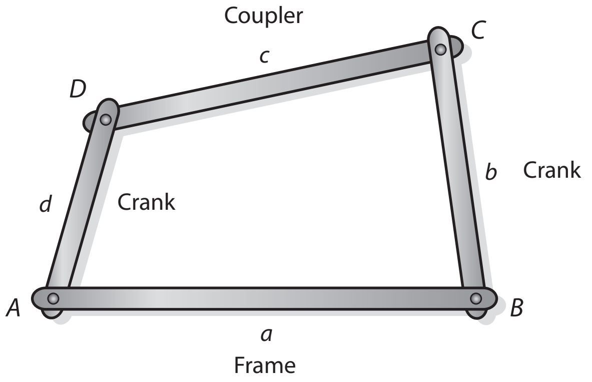
c. Now add a *diagonal* strip $\overline{BD}$ to your quadrilateral model. What features of this model can change?

d. Triangles are **rigid**. They retain their shape when pressure is applied. Quadrilaterals are rigid when *triangulated* with a diagonal. The process of triangulating is often called *bracing*. How are these facts utilized in the design of the bridge truss shown on page 363?

e. Describe two structures or objects in your community or home that employ the rigidity of triangles in their design.

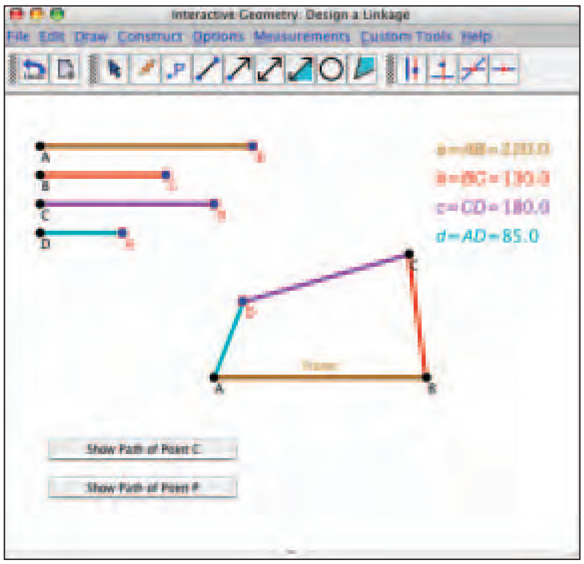
The nonrigidity of quadrilaterals has important physical applications. For example, mechanical engineers use the flexibility of quadrilaterals in the design of *linkages*.

6 An important feature of a quadrilateral or 4-bar linkage is that if any side is held fixed so it does not move and another side is moved, then the movement of the remaining sides is completely determined. The side that is fixed is called the *frame*. The two sides attached to the frame are called *cranks*. The crank most directly affected by the user is called the *driver* crank; the other is called the *follower* crank. The side opposite the frame is called the *coupler*.



Quadrilateral linkages have different characteristics depending on the lengths a, b, c, d of the sides and which side is used as a crank. Explore some of those characteristics using linkage strips or computer software like the "Design a Linkage" custom tool.

a. Working with a partner, make several different quadrilateral linkages so that strip $\overline{AB}$ is the longest side and fixed; strip $\overline{AD}$ is the shortest side and acts as one of the cranks. Investigate how lengths a and d are related to lengths b and c when $\overline{AD}$ can rotate completely. In this case, how does the follower crank move? The coupler? Write a summary of your findings.



b. The principle you discovered in Part a is called **Grashof's Principle**. How could you use a mechanism satisfying Grashof's Principle to drive the agitator in a washing machine or an automotive windshield wiper?

c. Use a quadrilateral linkage satisfying Grashof's Principle to investigate the mechanics of the linkage if the shortest side is used as the frame. Summarize your findings.

7 Examine the bus windshield wiper mechanism shown at the left. The wiper blade is attached to the mechanism in a fixed position.

a. Make a sketch of this mechanism. Label the frame, cranks, and coupler.

b. Explain why this is a parallelogram linkage.

c. As the linkage moves, what paths do the ends of the wiper blade follow?

d. If the wiper blade is vertical (as shown) when the mechanism is at the beginning of a cycle, describe the positions of the blade when the mechanism is one quarter of the way through its cycle and when the mechanism is halfway through its cycle.

e. Sketch the region of the windshield that the blade can keep clean.

Summarize the Mathematics

In this investigation, you experimented with building triangles and quadrilaterals with different side lengths. You also investigated how the rigidity of triangles and the nonrigidity of quadrilaterals influence their uses in the design of structures and devices.

a Describe the similarities and differences in what you discovered in your triangle-building and quadrilateral-building experiments.

b Suppose you are told that a triangular garden plot is to have sides of length 5 m, 12 m, and 13 m.

- **i.** Explain why it is possible to have a triangular plot with these dimensions.
- **ii.** Explain how you and a partner could lay out such a plot using only a 15-meter tape measure.
- **iii.** How many differently-shaped triangular plots could be laid out with these dimensions? Why?

c What constraints are needed on the lengths of the sides of a quadrilateral for it to be a parallelogram? What additional constraint(s) are needed for it to be a rectangle?

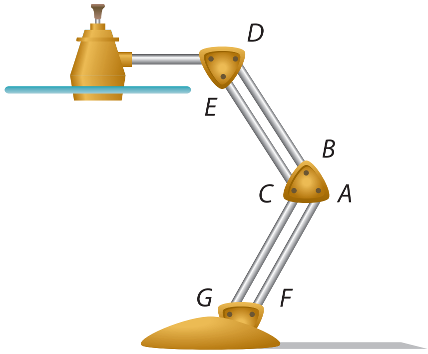
d What does it mean to say that a shape is rigid? How can you make a quadrilateral rigid?

e What must be true about the sides of a quadrilateral linkage if one of the cranks can make a complete revolution? If both cranks can make complete revolutions?

Be prepared to share your ideas and reasoning with the class.

Check Your Understanding

Four-bar linkages illustrate how geometric shape and function are related.



a. Examine the adjustable lamp in the diagram. The pivots at the labeled points are snug, but they will allow pivoting to adjust the lamp.

  i. Explain why the parallelogram linkages used in this lamp remain parallelograms as the position of the lamp is adjusted.

  ii. Visualize and describe how the position of the lamp should change as you make parallelogram *AFGC* vertical.

b. Suppose you are given segments of the following lengths: 7, 8, 24, 25.

  i. If possible, sketch and label several different quadrilaterals that can be formed with these side lengths.

  ii. Suppose you build a quadrilateral linkage with consecutive sides of lengths 7, 24, 8, and 25. What can you say about the length of the shortest possible brace that will make the quadrilateral rigid?

  iii. Can a quadrilateral linkage with a rotating crank be constructed from strips of these lengths? Explain your reasoning.

  iv. Can a quadrilateral linkage be made from these strips with two rotating cranks? Explain.

Investigation 2 Congruent Shapes

Roof trusses are manufactured in different shapes and sizes but they are most often triangular in shape. The "W" or Fink truss shown below is the most widely-used design in building today. The locations of the truss components provide for the most uniform distribution of stresses and forces. The rigidity of triangles is a key element in the design of these trusses. An equally important element is that all trusses for a particular roof are identical or *congruent*.

Teri Ziebarth

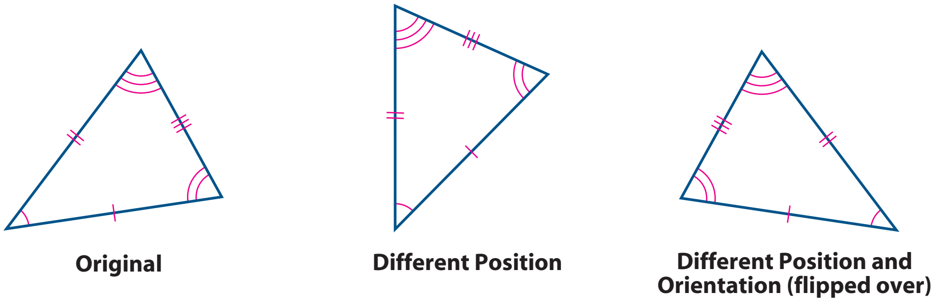
As you work on the problems of this investigation, look for answers to the following questions:

How can you test whether two shapes are congruent?

What combination of side or angle measures is sufficient to determine if two trusses or other triangular shapes are congruent?

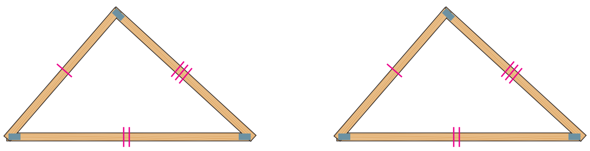
1 As a builder at the home site pictured on page 369, how could you test whether the two trusses standing against the garage wall are congruent? Could you use the same method to test if those two trusses are congruent to the ones already placed in position on the double-car garage?

Congruent figures have the same shape and size, regardless of *position* or *orientation*. In congruent figures, corresponding segments have the same length and corresponding angles have the same measure. The marks in the diagrams below indicate corresponding side lengths and angle measures that are identical.



One way to test for congruence of two trusses, or any two figures, is to see if one figure can be made to coincide with the other by sliding, rotating, and perhaps flipping it. This is, of course, very impractical for large trusses. Your work in the previous investigation suggests an easier method.

2 In Investigation 1, you found that given three side lengths that satisfy the Triangle Inequality, you could build only one triangle.



Side-Side-Side (SSS)

a. Explain as carefully as you can why simply measuring the lengths of the three corresponding sides of two triangular roof trusses is sufficient to determine if the trusses are congruent.

b. Could you test if the two trusses are congruent by measuring the lengths of just two corresponding sides? Explain.

In the following problems, you will explore other combinations of side lengths and angle measures that would provide a simple test of whether two triangular roof trusses are congruent.

3 Use strands of spaghetti along with a ruler and a compass, or geometry software like the "Triangle Congruence" custom tool to conduct the following triangle-building experiments.

For each condition in Parts a–c:

- Try to build a triangle satisfying the given condition. You choose segment lengths. Use one or two of the angles below as a template for the angle(s) of your triangle.

- If a triangle can be built, try to build another with the same three parts.
- Make a note if the condition could be used to test for congruence of two triangles.

For each experiment, compare your findings with your classmates and resolve any differences. Keep a record of your agreed-upon findings. Include sketches of the shapes you make.

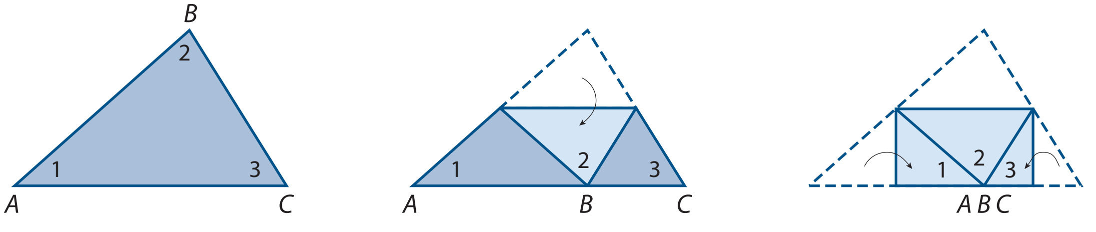
a. *Side-Angle-Side* (*SAS*) *Condition*: You know the lengths of two sides and the measure of the angle between the two sides.

b. *Side-Side-Angle* (*SSA*) *Condition*: You know the lengths of two sides and the measure of an angle not between the two sides.

c. *Angle-Side-Angle* (*ASA*) *Condition*: You know the measures of two angles and the length of the side between the two angles.

4 You may recall from your prior mathematics study that the sum of the measures of the angles of a triangle is 180°.

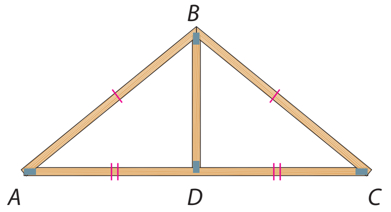
a. How is this **Triangle Angle Sum Property** demonstrated by folding a paper model of a triangle as shown below?



b. Using a protractor and ruler, carefully draw a triangle with angle measures 40°, 60°, and 80°.

c. Could a building contractor test whether two triangular roof trusses are congruent by measuring only the corresponding angles? Explain your reasoning.

5 The Kingpost truss shown below is used primarily for support of single-car garages or short spans of residential construction. The shape of the truss is an isosceles triangle. The support brace $\overline{BD}$ connects the peak of the truss to the midpoint of the opposite side.



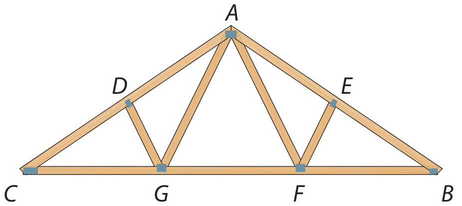
a. How are the specifications (given information) for this truss shown in the diagram?

b. Based on the specifications for this truss and the results of your experiments, explain as carefully as you can why $\triangle ABD$ is congruent to $\triangle CBD$, written $\triangle ABD \cong \triangle CBD$. (The congruence notation always lists the letters for corresponding vertices in the same order.)

c. To properly support the roof, it is important that the brace $\overline{BD}$ is perpendicular to side $\overline{AC}$. Based on your work in Part b, explain why the placement of brace $\overline{BD}$ guarantees that $\overline{BD}$ is perpendicular to $\overline{AC}$ (in symbols, $\overline{BD} \perp \overline{AC}$).

d. An important property of the Kingpost truss, and *any* isosceles triangle, is that the angles opposite the congruent sides (called **base angles**) are congruent. How does your work in Part b guarantee that $\angle A \cong \angle C$?

6 Study the diagram below of a "W" truss. $\triangle ABC$ is an isosceles triangle. Points D, E, F, and G are marked on the truss so that $\overline{CG} \cong \overline{BF}$ and $\overline{CD} \cong \overline{BE}$.



a. On a copy of the truss, use tick marks to show the given information.

b. When building the truss, explain as carefully as you can why braces $\overline{DG}$ and $\overline{EF}$ should be cut the same length.

c. Should braces $\overline{AG}$ and $\overline{AF}$ be cut the same length? Explain your reasoning.

Summarize the Mathematics

In this investigation, you discovered combinations of side lengths or angle measures that were sufficient to determine if two triangles were congruent. You also explored how you could use congruent triangles to reason about properties of an isosceles triangle.

a Which sets of conditions—SSS, SAS, SSA, ASA, and AAA—can be used to test if two triangles are congruent?

b Write each *Triangle Congruence Condition* in words and illustrate with a diagram.

c If $\triangle PQR \cong \triangle XYZ$, what segments are congruent? What angles are congruent?

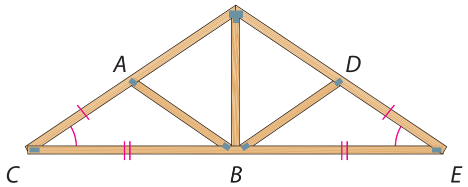
d Describe properties of an isosceles triangle that you know by definition or by reasoning.

Be prepared to share your ideas and reasoning with the class.

Check Your Understanding

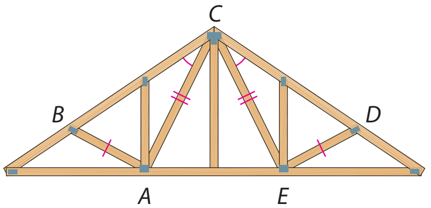
Wood trusses commonly employ two or more triangular components in their construction. For each truss below, examine the two labeled triangular components. Is enough information provided for you to conclude that the triangles are congruent? Explain your reasoning.

a.



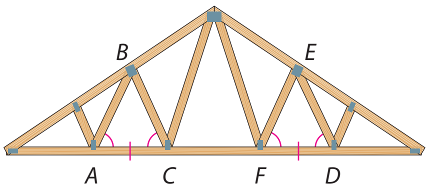
Queenpost Truss

b.



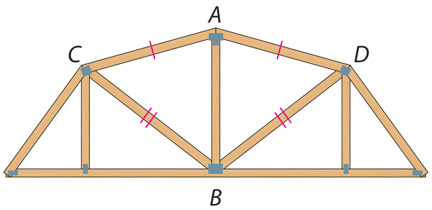
Fan Truss

c.



Double Fink Truss

d.



Gambrel Truss

Investigation 3 Reasoning with Shapes

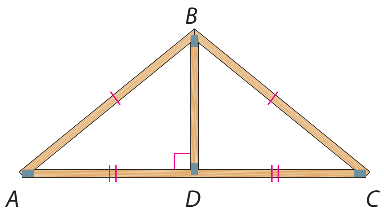
In your work with the Kingpost truss, you discovered some important properties of isosceles triangles—not by conducting experiments and looking for patterns but by careful reasoning from statements of facts that you and your classmates already understand and agree on. As you work on problems of this investigation, look for answers to the following questions:

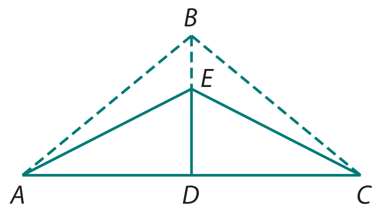
What strategies are useful in reasoning about properties of shapes?

What are some additional properties of triangles and quadrilaterals that have important applications?

1 As you may recall, the support brace $\overline{BD}$ of a Kingpost truss as shown below on the left connects the peak of the truss to the midpoint of the opposite side. You used congruent triangles to show that $\overline{BD} \perp \overline{AC}$. In this case, $\overline{BD}$ is said to be a **perpendicular bisector** of $\overline{AC}$, that is $\overline{BD} \perp \overline{AC}$ at the midpoint D of $\overline{AC}$.

To design a Kingpost truss that has the same *span* $\overline{AC}$, but less *pitch* (slope), Beth located point E on the perpendicular bisector of $\overline{AC}$ as shown below on the right.





She was confident that the new truss would still be an isosceles triangle. She reasoned as follows:

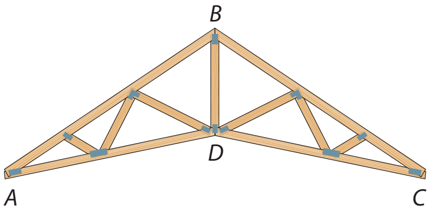
I need to show that $\overline{EA} \cong \overline{EC}$. Consider $\triangle ADE$ and $\triangle CDE$.
Since $\overline{ED}$ is the $\perp$ bisector of $\overline{AC}$, $\angle ADE$ and $\angle CDE$ are right angles and $\overline{AD} \cong \overline{CD}$. The triangles share $\overline{ED}$.
So, $\triangle ADE \cong \triangle CDE$.
Since corresponding parts of congruent triangles are congruent, $\overline{EA} \cong \overline{EC}$.

a. Is Beth's reasoning correct? How does she know that $\triangle ADE \cong \triangle CDE$?

b. On a copy of the diagram above on the left, design a new truss that has the same span but greater pitch by locating a point F on the line $\overleftrightarrow{BD}$. Explain carefully why your truss is an isosceles triangle.

c. Explain why *any* point on the perpendicular bisector of a segment will be equally distant from the endpoints of the segment.

2 The truss shown at the right is often used for portions of a house in which a sloped interior ceiling is desired. It is designed so that $\overline{AB} \cong \overline{CB}$ and $\overline{AD} \cong \overline{CD}$.



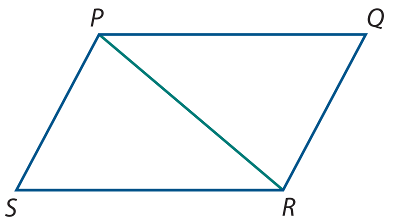
Modified Queen Scissors Truss

a. How could you reason with congruent triangles to explain why $\angle ABD \cong \angle CBD$?

b. What other pairs of angles in the truss must also be congruent? Why?

In Investigation 1, you found that you could make a quadrilateral linkage rigid by adding a diagonal brace. Diagonals are also helpful in reasoning about properties of quadrilaterals.

3 Recall that by definition of a parallelogram, opposite sides are the same length, or congruent.



a. On a copy of parallelogram *PQRS*, use tick marks to indicate segments that are congruent.

b. Provide an argument to justify the statement:

A diagonal of a parallelogram divides the parallelogram into two congruent triangles.

c. Angles in a parallelogram like $\angle Q$ and $\angle S$ are called **opposite angles**.

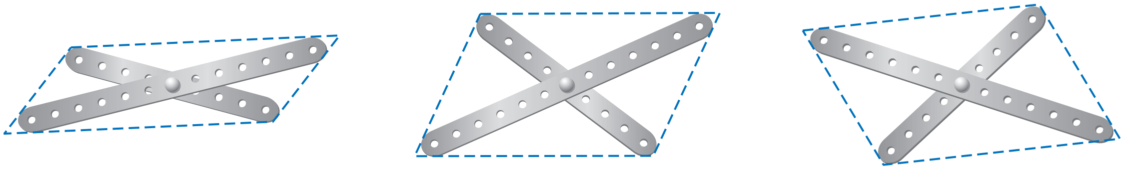
i. Explain why $\angle Q \cong \angle S$.

ii. What reasoning would you use to show that the other pair of opposite angles, $\angle P$ and $\angle R$, are congruent? Compare your argument with others.

d. What is the sum of the measures of the angles of $\square PQRS$? Give reasons that support your answer.

e. Would your answer and reasons in Part d change if the figure were a quadrilateral but *not* a parallelogram? Explain your reasoning.

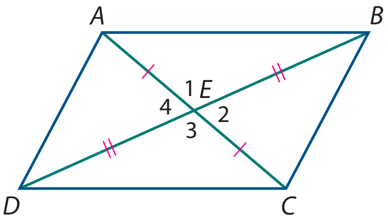
4 Information on diagonal lengths can be used to test whether a quadrilateral is a special quadrilateral. The diagram below shows results of three trials of an experiment with two linkage strips fastened at their midpoints.



a. In each case, what appears to be true about the quadrilateral that has the given strips as its diagonals? Do you think the same conclusion would hold if you conducted additional trials of the experiment?

Parts b–e will provide you a guide to preparing a supporting argument for the statement:

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.



b. Study this diagram of a quadrilateral with diagonals that bisect each other.

i. Are pairs of segments given as congruent properly marked? Explain.

ii. To show quadrilateral $ABCD$ is a parallelogram, you must show that opposite sides are the same length. To show that opposite sides $\overline{AB}$ and $\overline{CD}$ are congruent, what triangles would you try to show are congruent? What additional information would you need?

c. Angles positioned like $\angle 1$ and $\angle 3$, and $\angle 2$ and $\angle 4$, are called **vertical angles**. Each pair of vertical angles *appears* to be congruent. A student at Bellevue High School in Washington gave the following argument to justify that $\angle 1 \cong \angle 3$.

i. Give a reason to support each statement.

$m\angle 1 + m\angle 2 = 180°$	($m\angle 1$ is read "measure of $\angle 1$")	(1)
$m\angle 2 + m\angle 3 = 180°$		(2)
$m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$		(3)
$m\angle 1 = m\angle 3$		(4)
So, $\angle 1 \cong \angle 3$.		(5)

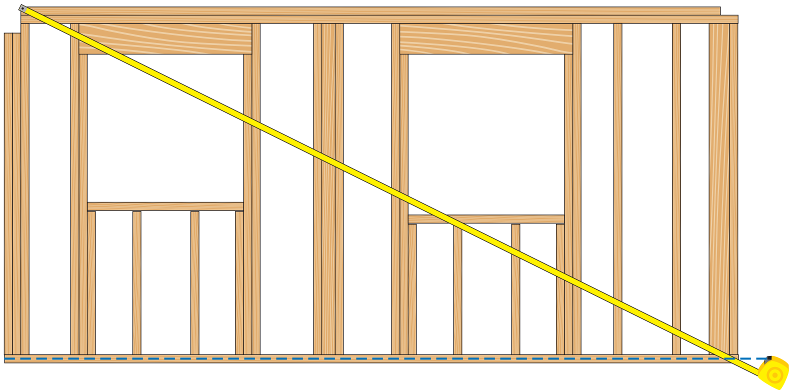
ii. Use similar reasoning to write an argument justifying that $\angle 2 \cong \angle 4$.

d. Explain why it follows that $\triangle AEB \cong \triangle CED$ and $\triangle AED \cong \triangle CEB$.

e. Why can you conclude that $\overline{AB} \cong \overline{CD}$ and $\overline{AD} \cong \overline{CB}$? That quadrilateral $ABCD$ must be a parallelogram?

5 Diagonal lengths are frequently used in "squaring" building foundations and setting walls in the construction of homes. To square a wall, the bottom plate is held secure and the top of the wall is adjusted until both diagonal measures are the same.

Matt Meadows



a. Assuming the top and bottom plates are the same length and the two wall studs at the ends are the same length, explain as carefully as you can why the statement, "If the diagonals are the same length, then the wall frame is a rectangle," is true. Your explanation should include a labeled diagram, a statement of what information is given in terms of the diagram, and supporting reasons for your statements.

b. Compare your argument with others. Correct any errors in reasoning.

Summarize the Mathematics

In this investigation, you used Triangle Congruence Conditions to support your reasoning about properties of shapes.

a What is true about any point on the perpendicular bisector of a segment? How is this related to congruence of triangles?

b What is the sum of the measures of the interior angles of any quadrilateral? How could you convince others of this property?

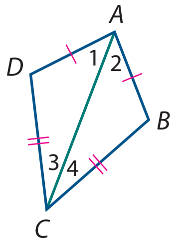
c What are some special properties of parallelograms? Of rectangles? How are these properties related to congruence of triangles?

d What are some general strategies to consider when trying to establish properties of shapes by reasoning?

Be prepared to share your ideas with the class.

Check Your Understanding

Refer to kite *ABCD* with diagonal $\overline{AC}$ shown at the right.



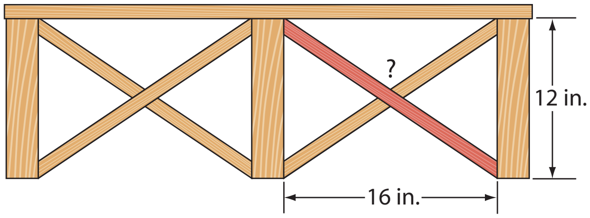
a. Use careful reasoning to explain why $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$.

b. What must be true about the shorter diagonal $\overline{DB}$? Why?

Investigation 4 Getting the Right Angle

Your work on problems in the previous investigations illustrated three important aspects of doing mathematics—experimenting, reasoning from accepted facts to new information, and applying those ideas to practical problems. In your triangle-building experiments, you discovered patterns that suggested the reasonableness of the Triangle Inequality and the Triangle Congruence Conditions. Using various congruence conditions, you were able to carefully reason to properties of special triangles and quadrilaterals. You then applied those properties to a variety of problems. Keep these aspects of doing mathematics in mind as you complete this investigation.

1 Bridging, shown in the diagram below, provides stability between adjacent floor joists. It is generally used when floor spans are greater than 8 feet. If the floor joists are set approximately 16 inches apart, to what length should the bridging be cut? Why should all pieces be cut the same length?



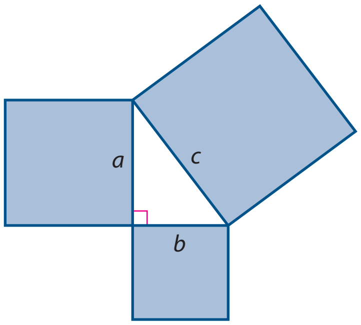
In working on Problem 1, you likely used a special property of right triangles—the Pythagorean Theorem. Your work on the remaining problems of this investigation will help you answer these questions:

Why is the Pythagorean Theorem true for all right triangles?
Is the converse of the Pythagorean Theorem true and, if so, why?

The Pythagorean Theorem is often used to calculate the length of the hypotenuse of a right triangle. You can also think of the Pythagorean Theorem as a statement of a relationship among areas of three squares.

For any right triangle, the area of the square built on the hypotenuse is equal to the sum of the areas of the squares built on the two legs.

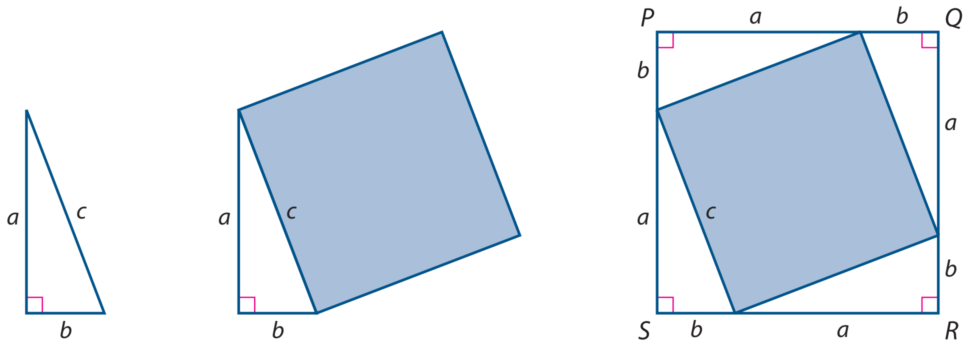
$$a^2 + b^2 = c^2$$



The Greek philosopher Pythagoras (572–497 B.C.) is sometimes credited with first providing a general argument for why this relationship is true for all right triangles. However, the oldest recorded justification is found in an ancient Chinese manuscript written more than 500 years before Pythagoras. The ancient Babylonians and Egyptians also discovered special cases of the relationship.

Since there are infinitely many right triangles, it would be impossible to check that $a^2 + b^2 = c^2$ for all of them. Pythagoras's argument, like that outlined in Problem 2, involves reasoning from known facts rather than relying on patterns in specific right triangles. In Problem 2, the argument involves finding the area of the same square *PQRS* in two different ways.

2 Study the diagrams below of a right triangle, a square built on the hypotenuse of the triangle, and an arrangement of congruent copies of the triangle around the square.

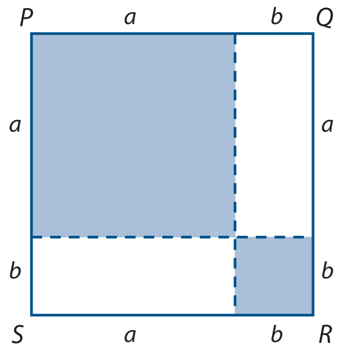


a. Explain as carefully as you can why quadrilateral *PQRS* is a square. Your explanation should include how you know that the sides are straight line segments.

b. Describe two ways to calculate the area of square *PQRS*.

c. Now study this diagram, which shows another way of thinking about the area of square *PQRS*.

On a copy of this diagram, add two line segments to create four right triangles congruent to the original right triangle. Explain how you know that the triangles are congruent.



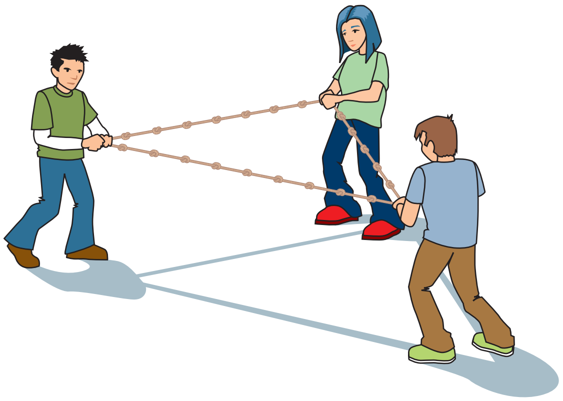
d. Place a copy of the right-most diagram above Part a side-by-side with your modification of the diagram in Part c.

i. How do the areas of the two large squares compare?

ii. Suppose you remove the four congruent triangles from each of the diagrams. What can you say about the areas of the remaining pieces?

iii. Explain as precisely as you can what you have shown.

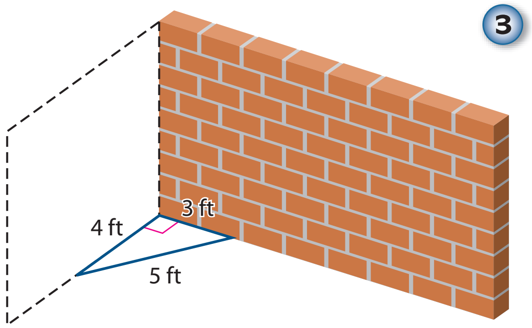
Now look back at the rope-stretching problem at the beginning of this lesson (page 363). In an attempt to form a right triangle, one group of students at Washington High School stretched the knotted rope as shown below.



They claimed the triangle was a right triangle since $8^2 + 6^2 = 10^2$. These students used the **converse of the Pythagorean Theorem** in their reasoning:

If the sum of the squares of the lengths of two sides of a triangle equals the square of the length of the third side, then the triangle is a right triangle.

The **converse** of an *if-then* statement reverses the order of the two parts of the statement. Although the converse of the Pythagorean Theorem *is* true, the converse of a true statement may not necessarily be true. For example, consider the statement, "If I'm in math class, then I'm in school," and the converse, "If I'm in school, then I'm in math class." Is the converse necessarily true?



3 To lay out a wall perpendicular to an existing wall, a builder measures 3 feet along the base of the existing wall and 4 feet along the floor line where the new wall is to be placed. The builder then checks if the distance between these two points is 5 feet. If so, she knows that the angle between the existing wall and the wall to be constructed is 90°.

a. Is the builder using the Pythagorean Theorem or the converse of the Pythagorean Theorem? Explain.

b. You can use your understanding of triangle congruence to explain why this "3-4-5 triangle" method guarantees a right angle.

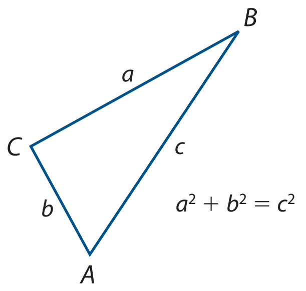
i. Draw segments of length 3 cm, 4 cm, 5 cm. Then, using a ruler and compass, construct a triangle with these side lengths.

ii. Use a ruler and protractor to draw a separate 90° angle. From the vertex of the angle, mark off a segment of length 3 cm on a side and of 4 cm on the other side. Connect the two sides to form a right triangle. According to the Pythagorean Theorem, what should be the length of the hypotenuse?

iii. Explain why the 3-4-5 triangle in part i is congruent to the triangle in part ii.

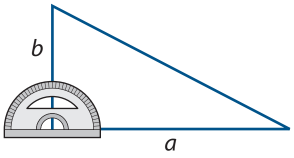
iv. Why must the 3-4-5 triangle have a right angle? Where is it located?

c. You can use similar reasoning to show, in general, that if you start with a $\triangle ABC$ where the lengths of its sides a, b, and c satisfy $a^2 + b^2 = c^2$, then you can conclude that $\triangle ABC$ is a right triangle with right angle at C.



To prove that $\triangle ABC$ is a right triangle, you can reason like you did in Part b.

i. On a separate sheet of paper, draw and label a *right* triangle with sides (other than the hypotenuse) of the given lengths a and b.

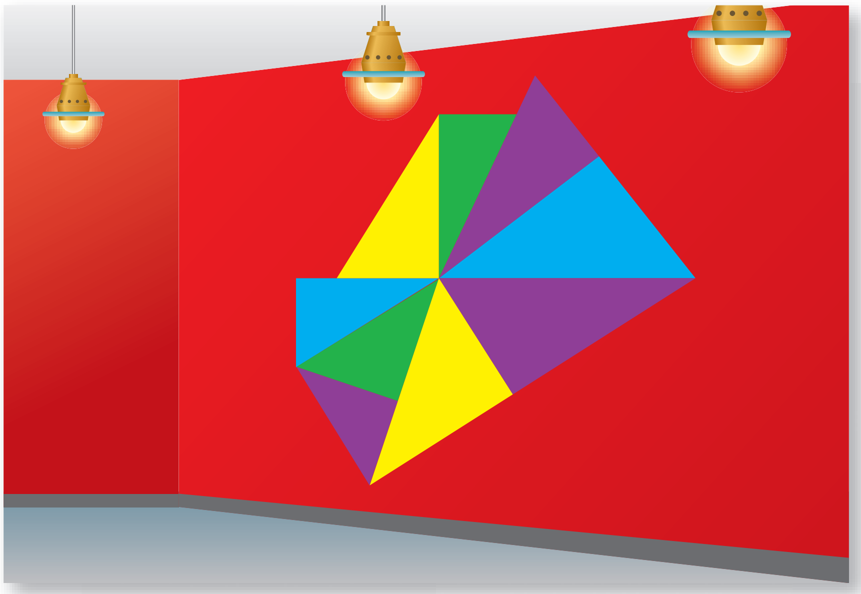


ii. Write an expression for the length of the hypotenuse of this triangle.

iii. Why is the triangle you created congruent to the given triangle, $\triangle ABC$?

iv. Why must the given triangle, $\triangle ABC$, be a right triangle? Why is $\angle C$ the right angle?

4 In preparing a wall design using triangles, is it possible to draw a triangle congruent to a given *right* triangle under each of the following conditions? In each case, explain your reasoning.



a. You measure the lengths of the two legs of the given right triangle.

b. You measure the lengths of a leg and the hypotenuse of the given right triangle.

Summarize the Mathematics

In this investigation, you examined applications of the Pythagorean Theorem and its converse. You also used careful reasoning to provide arguments for why these statements are true.

a. Describe the general idea behind your argument that the Pythagorean Theorem is true for all right triangles.

b. Describe the general idea behind your argument that the converse of the Pythagorean Theorem is true.

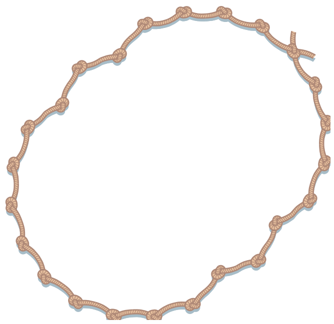
c. Give two examples, one mathematical and one not involving mathematics, to illustrate that if a statement is true, its converse may not be true.

d. What is the smallest number of side lengths you need to compare in order to test if two right triangles are congruent? Does it make a difference which side lengths you use? Explain.

Be prepared to share ideas and examples with the class.

Check Your Understanding

In the Think About This Situation (page 363), you were asked to consider whether four students could form various shapes using a 24-meter loop of knotted rope with knots one meter apart. Reconsider some of those questions using the mathematics you learned in this investigation.



a. Explain how you could use the 24-meter knotted rope to form a right triangle and how you know the shape is a right triangle.

b. Now explain how you could use the 24-meter knotted rope to form a rectangle and how you know that the shape is a rectangle.

c. Look back at your work in Part b. Could you form a second differently shaped rectangle? Explain.

d. Suppose you and two classmates were given a 30-meter loop of rope with knots tied one meter apart. Could you position yourselves so that the resulting triangle is a right triangle? Explain your reasoning.

On Your Own

Applications

1 Suppose you are given four segments with lengths 5 cm, 5 cm, 12 cm, 12 cm. Think about building shapes using three or four of these lengths.

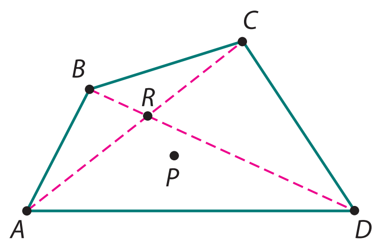
a. How many different triangles can you build? Identify any special triangles.

b. Can you build a parallelogram? If so, how many different ones can you build?

c. Can you build a kite? If so, how many different kites can you build?

d. How many different quadrilaterals can you build that are not parallelograms?

2 Four large oil fields are located at the vertices of a quadrilateral $ABCD$ as shown. Oil from each of the four fields is to be pumped to a central refinery. To minimize costs, the refinery is to be located so that the amount of piping required is as small as possible.



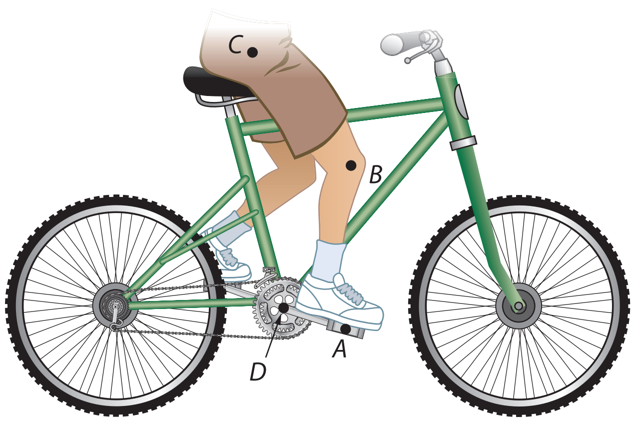
a. If the refinery is located at position R, write an algebraic expression that shows the amount of piping required.

b. For oil fields A and C, explain why position R is a better location for the refinery than position P.

c. Explain why position R is a better location for the refinery than position P in terms of all four oil fields.

d. Is there a better location for the refinery than position R? Explain your reasoning.

3 Understanding the body mechanics involved in various physical activities is important to sports physicians and trainers. The diagram at the right shows a person pedaling a bicycle. Key points in the pedaling motion are labeled.



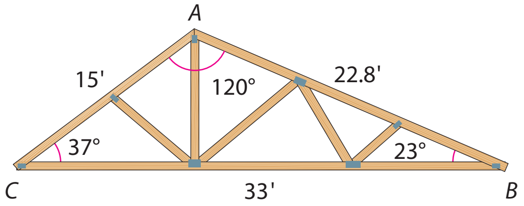
a. What kind of linkage is represented by $ABCD$?

b. Identify the frame, the coupler, the drive crank, and the follower crank.

c. What modifications to the situation would allow it to be modeled by a parallelogram linkage? Should a sports trainer recommend these modifications? Explain your reasoning.

On Your Own

4 A Double Pitch truss, with side lengths and angle measures, is shown below.



Double Pitch Truss

Which sets of measurements below would be sufficient to test whether a truss PQR is congruent to the given truss ABC? Explain your reasoning in each case.

a. $PQ = 22.8'$, $PR = 15'$, and $m\angle P = 120°$

b. $PQ = 22.8'$, $PR = 15'$, and $m\angle R = 23°$

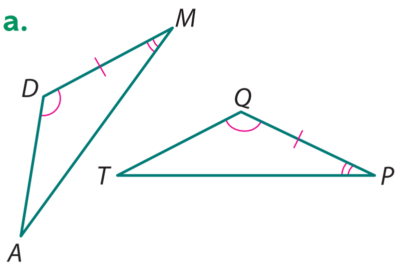
c. $RQ = 33'$, $m\angle Q = 23°$, and $m\angle R = 37°$

d. $m\angle P = 120°$, $m\angle R = 37°$, and $m\angle Q = 23°$

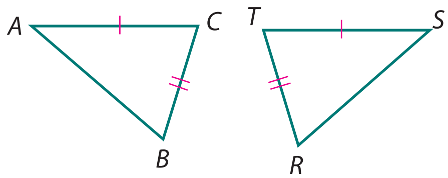
e. $PQ = 22.8'$, $RQ = 33'$, and $PR = 15'$

5 Examine each of the following pairs of triangles and the markings that indicate congruence of corresponding angles and sides. In each case, decide whether the information given by the markings ensures that the triangles are congruent. If the triangles are congruent, write the congruence relation and cite an appropriate congruence condition to support your conclusion.

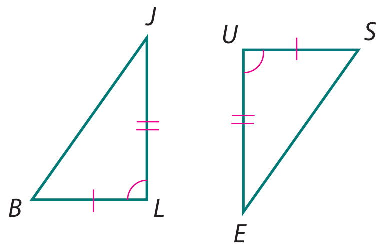
a.



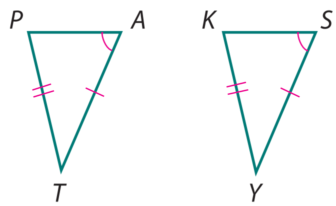
b.



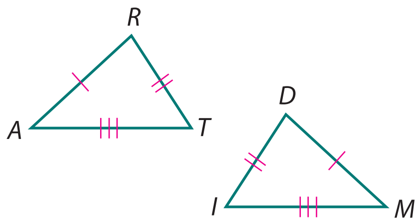
c.



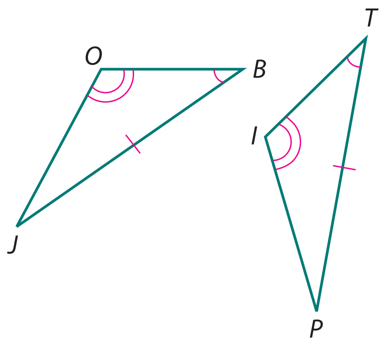
d.



e.

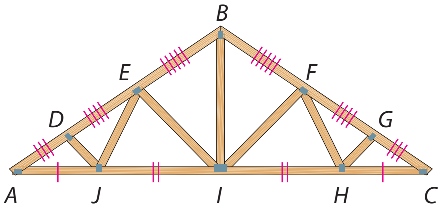


f.



6 Modified Queenpost trusses are often used for roofs that have wide spans and low pitch.

In manufacturing this particular isosceles triangular truss, the bracing is positioned according to specifications in the diagram below.



Modified Queenpost Truss

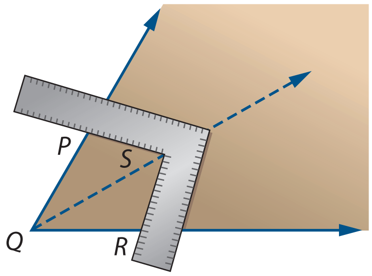
a. Explain carefully why braces $\overline{DJ}$ and $\overline{GH}$ should be cut the same length.

b. Explain why braces $\overline{EJ}$ and $\overline{FH}$ should be cut the same length.

c. Give reasons why $\overline{EI}$ and $\overline{FI}$ should be cut the same length.

d. Is quadrilateral *EBFI* a special quadrilateral? If so, name it and explain how you know.

7 The diagram below illustrates how a carpenter's square is often used to bisect an angle. (A **bisector of an angle** is a ray that begins at the vertex of the angle and divides the angle into two angles of equal measure.) The square is positioned as shown so that $PQ = RQ$ and $PS = RS$.



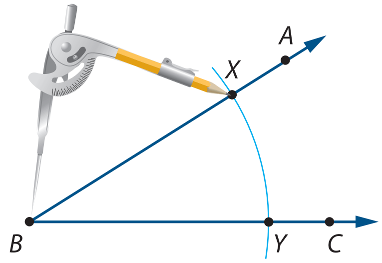
a. Explain why this information is sufficient to conclude that $\triangle PQS \cong \triangle RQS$.

b. Why does ray *QS* (written $\overrightarrow{QS}$) bisect $\angle PQR$?

On Your Own

8 Draftsmen and industrial designers use a variety of tools in their work. Depending on the nature of the task, these tools vary from sophisticated CAD (computer-assisted design) software to compasses and *straightedges* (rulers with no marks for measuring).

a. Draw an acute angle, $\angle ABC$. Using a compass, a straightedge, and the algorithm below, construct the bisector of $\angle ABC$.



Angle Bisector Algorithm: To bisect $\angle ABC$, do the following.

Step 1: With the compass point at B, draw an arc that intersects $\overrightarrow{BA}$ and $\overrightarrow{BC}$; call the intersection points X and Y, respectively.

Step 2: With the compass point at point X and using a radius greater than $\frac{1}{2}XY$, draw an arc in the interior of $\angle ABC$. Then, keeping the same radius, place the compass point at Y and draw a second arc that intersects the first. Label the point of intersection D.

Step 3: Draw the ray $\overrightarrow{BD}$. $\overrightarrow{BD}$ bisects $\angle ABC$.

b. Explain why this algorithm produces the bisector of $\angle ABC$. That is, explain how you know that $\overrightarrow{BD}$ bisects $\angle ABC$. In what way(s) is this algorithm similar to the technique in Applications Task 7?

c. Can this algorithm be used to construct the bisector of a right angle and an *obtuse angle* (an angle with measure greater than 90°)? Explain your reasoning.

d. Think of a line as a "straight" angle. Add steps to the Angle Bisector Algorithm to produce an algorithm for constructing a perpendicular to a given point P on a line.

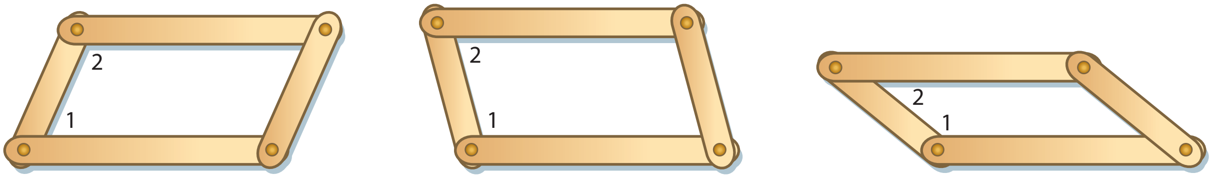
i. Draw a line $\overleftrightarrow{AB}$ containing point P. Use your algorithm and a compass and straightedge to construct a perpendicular to $\overleftrightarrow{AB}$ at P.

ii. Explain how you know that the line you constructed is perpendicular to $\overleftrightarrow{AB}$ at P.

e. How would you modify your algorithm to construct a perpendicular bisector of a segment? Explain as carefully as you can why your method works.

9 Use a ruler to carefully draw a triangle, $\triangle XYZ$. Design and test an algorithm for using a compass and a straightedge to construct $\triangle ABC$ so that $\triangle ABC \cong \triangle XYZ$. Provide an argument that your algorithm will always work.

10 In Investigation 3, you were able to provide an argument for why opposite angles of any parallelogram are congruent. Experimenting with a parallelogram linkage should convince you that **consecutive angles** of a parallelogram like $\angle 1$ and $\angle 2$ may not always be congruent.



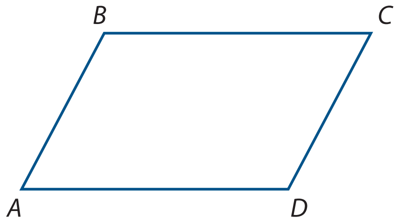
In the first diagram, $m\angle 1 < m\angle 2$. In the next two diagrams, as $m\angle 1$ increases, $m\angle 2$ decreases. Thinking that there might be some relationship between the angles, students in a class at Columbia-Hickman High School measured the angles and in each case found that $m\angle 1 + m\angle 2$ was about 180°. They tried to find reasons that might explain this relationship.

Examine the reasoning of each student below.

- Give a reason that would support each statement made by the students.
- Then decide if the conclusion follows logically from knowing that quadrilateral *ABCD* is a parallelogram.

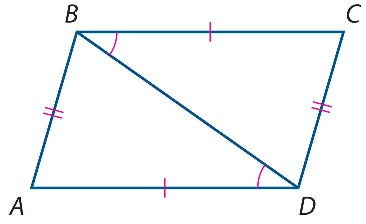
a. Anna drew $\square ABCD$ at the right and set out to show that $m\angle A + m\angle B = 180°$. She reasoned as follows.

Since *ABCD* is a quadrilateral, I know that
$m\angle A + m\angle B + m\angle C + m\angle D = 360°$.
Since *ABCD* is a parallelogram, I know that
$\angle A \cong \angle C$ and $\angle B \cong \angle D$.
It follows that $m\angle A + m\angle B + m\angle A + m\angle B = 360°$.
So, $2m\angle A + 2m\angle B = 360°$.
Therefore, $m\angle A + m\angle B = 180°$.



b. Andy drew $\square ABCD$ with diagonal $\overline{BD}$ and then reasoned to show that $m\angle A + m\angle B = 180°$.

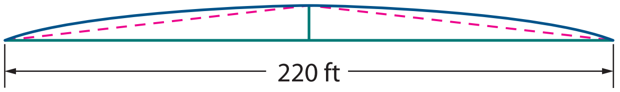
I know that $\triangle ABD \cong \triangle CDB$.
So, $\angle BDA \cong \angle DBC$.
I know that $m\angle A + m\angle ABD + m\angle BDA = 180°$.
So, $m\angle A + m\angle ABD + m\angle DBC = 180°$.
Therefore, $m\angle A + m\angle B = 180°$.



On Your Own

11 Materials tend to expand when heated. This expansion needs to be considered carefully when building roads and railroad tracks.

In the case of a railroad track, each 220-foot-long rail is anchored solidly at both ends. Suppose that on a very hot day a rail's length expands by 1.2 inches, causing it to buckle as shown below.



a. At what point along the rail do you think the buckling will occur?

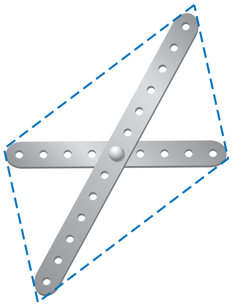
b. Do you think you could slide a gym bag between the raised rail and the track bed?

c. Approximate this situation using right triangles, and then calculate an estimate of the height of the buckle.

d. Would you expect your estimate of the height of the buckle to be more or less than the actual value? Explain your reasoning.

e. Research *expansion joints*. How does the use of these joints in railroad tracks and concrete highways minimize the problem you modeled in Part c?

12 You can represent the diagonals of a quadrilateral with two linkage strips attached at a point.



a. What must be true about the diagonal strips, and how should you attach them so that the quadrilateral is a parallelogram?

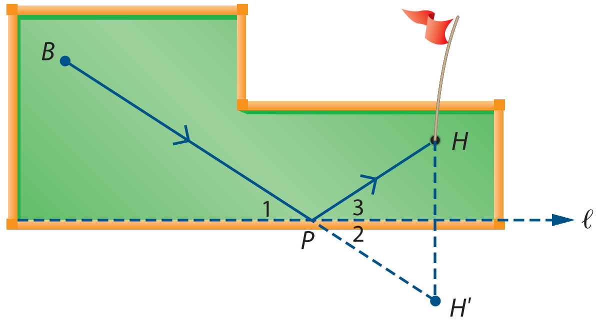
b. What must be true about the diagonal strips, and how should you attach them so that the quadrilateral is a rectangle?

c. What constraint(s) must be placed on the diagonal strips and their placement if the quadrilateral is to be a kite? Give reasons to justify that the shape with your arrangement of diagonals is a kite.

d. What constraints must be placed on the diagonal strips and how they are attached in Part a if the quadrilateral is to be a square? Give reasons to justify that the shape with your arrangement of diagonals is a square.

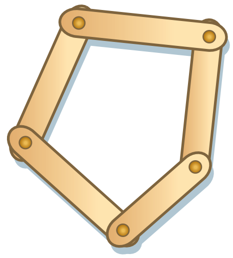
13 When a ball with no spin and medium speed is banked off a flat surface, the angles at which it strikes and leaves the cushion are congruent. You can use this fact and knowledge of congruent triangles to your advantage in games of miniature golf and pool.

To make a hole-in-one on the miniature golf green to the right, visualize a point H' so that the side ℓ is the perpendicular bisector of H and H'. Aim for the point P where $\overline{BH}$ intersects ℓ. If you aim for point P, give reasons to justify that the ball will follow the indicated path to the hole. That is, show that $\angle 3 \cong \angle 1$.



Connections

14 Examine the 5-bar linkage at the right.



a. Explain why this linkage is not rigid.

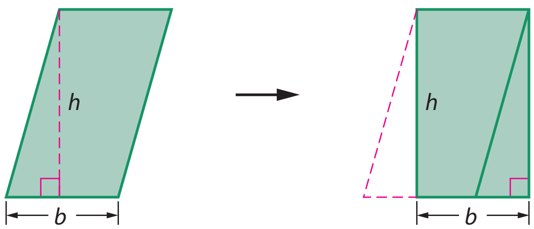
b. Make a sketch of the linkage showing how you could make it rigid. How many braces did you use? Is that the fewest number possible?

c. What is the fewest number of braces required to make a 6-bar linkage rigid? To make an 8-bar linkage rigid? Draw a sketch illustrating your answers.

d. Try to generalize your reasoning. What is the fewest number of braces required to make an n-bar linkage rigid? How many triangles are formed?

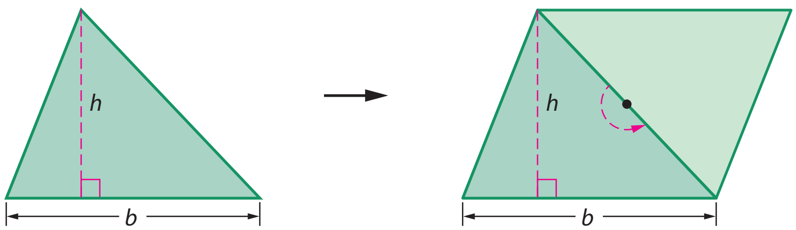
15 *Perimeter* and *area* are important characteristics of two-dimensional shapes. By recalling the formula for the area of a rectangle $A = base \times height$ and using visual thinking, you can develop and easily recall formulas for the areas of parallelograms and triangles. Study each pair of diagrams below in which b is the length of a *base* and h is the corresponding *height* of the shape. Write a formula for the area A of the shape and then explain how the diagrams helped you reason to the formula.

a.



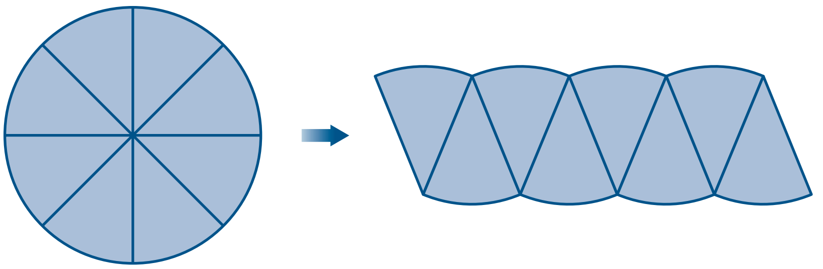
$A =$ ___?___

b.



$A =$ ___?___

16 The circle below has been dissected into eight sections. These sections can be reassembled to form an "approximate" parallelogram.



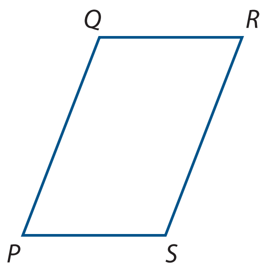
a. How is the base of this "approximate" parallelogram related to the circle?

b. What is the height of the "approximate" parallelogram?

c. How could you dissect the circle into sections to get a better approximation to a parallelogram?

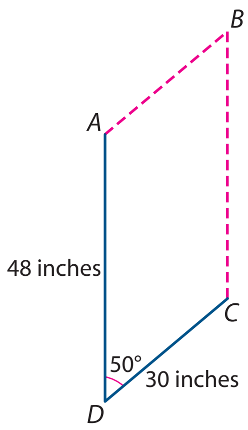
d. Use the above information to produce the formula for the area of a circle.

17 In Applications Task 10, you gave reasons why the sum of the measures of two consecutive angles of a parallelogram is 180°. In the case of ▱*PQRS*, this means that $m\angle P + m\angle Q = 180°$ and $m\angle P + m\angle S = 180°$.



a. Write two similar statements involving other pairs of angles of ▱*PQRS*.

b. Suppose you are given the indicated specifications for a parallelogram window frame *ABCD*.



i. Is this enough information to build the frame? Explain your reasoning.

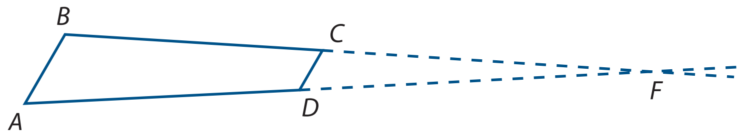
ii. Do you think there is an SAS condition for congruence of parallelograms? Explain.

Photo courtesy of Hurd Windows & Doors

18 The sum of the measures of consecutive angles of a parallelogram is 180°. (See Applications Task 10.) That property helps to explain the use of the term "parallel" in parallelogram.

In a parallelogram, opposite sides are parallel.

That is, if opposite sides of a parallelogram are extended, the lines will not intersect. Parts a–c provide an outline of why, in $\square ABCD$, $\overline{BC}$ must be parallel to $\overline{AD}$. The reasoning depends on you.



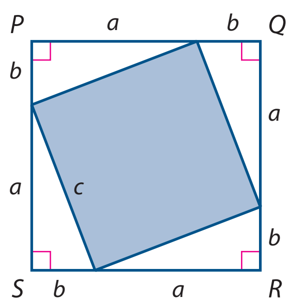
a. What is true about $m\angle A$ and $m\angle B$? Why?

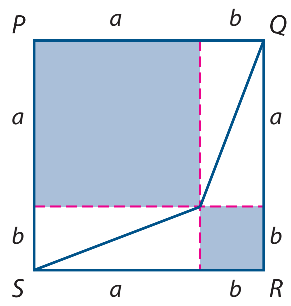
b. Now either $\overline{BC}$ is parallel to $\overline{AD}$, or $\overline{BC}$ is *not* parallel to $\overline{AD}$. If $\overline{BC}$ is *not* parallel to $\overline{AD}$, then the situation would look something like that in the diagram above on the right. What must be true about $m\angle A + m\angle B + m\angle F$? Why?

c. Explain why the situation in Part b is impossible. What does this tell you about the assumption that $\overline{BC}$ was not parallel to $\overline{AD}$? What can you conclude?

d. How could you use similar reasoning to show that $\overline{AB}$ must be parallel to $\overline{CD}$?

19 Two diagrams used in your reasoning about the Pythagorean Theorem (page 379) are shown below.





a. Write an expression for the area of square *PQRS* that involves a, b, and c using the diagram above at the left.

b. Write an algebraic expression for the area of square *PQRS* that involves only a and b, using the diagram above at the right.

c. Use your results from Parts a and b and algebra to show that $a^2 + b^2 = c^2$.

On Your Own

20 Draw squares of side lengths 2, 4, 7, 8, 10, and 11 centimeters on centimeter grid paper.

a. Measure the diagonals to the nearest 0.1 cm. Record your data in a table.

b. Make a plot of your (*side length, diagonal length*) data. Find a linear model that fits the trend in the data.

i. What is the slope of the line? What does it mean?

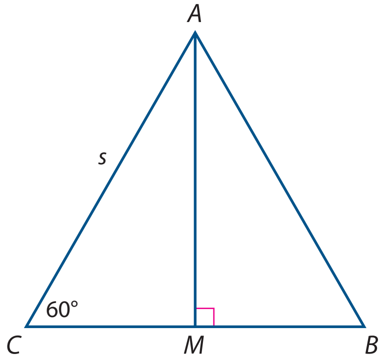
ii. What is the y-intercept? Does it make sense in this context? Explain.

c. Use your model to predict the length of the diagonal of a square with side length of 55 cm.

d. Compare your predicted length to that computed by using the Pythagorean Theorem. Explain any differences.

e. Write a rule that would express *exact* diagonal length D in terms of side length s for any square.

21 The diagram below shows an equilateral triangle, $\triangle ABC$, with an altitude from A to M that forms two smaller triangles.



a. What is the measure of $\angle MAC$?

b. Explain as precisely as you can why $\triangle AMC \cong \triangle AMB$.

c. Find the exact length of the altitude $\overline{AM}$ when the sides of the equilateral triangle have length:

i. 5 cm **ii.** 8 cm

iii. 10 cm **iv.** 1 cm

d. Now consider the general problem of finding side lengths of a 30°-60°-90° triangle in which the hypotenuse is of length s.

i. What formula gives the length of the side opposite the 30° angle?

ii. What formula gives the length of the side opposite the 60° angle?

e. Write in words how the lengths of the sides of a 30°-60°-90° triangle are related.

Reflections

22 In Investigation 3, you were able to provide reasons justifying that the base angles of an isosceles triangle are congruent. Why does it follow logically that an equilateral triangle is *equiangular*; that is, the three angles of an equilateral triangle are congruent? What is the measure of each angle?

23 Why will any parallelogram linkage have two rotating cranks?

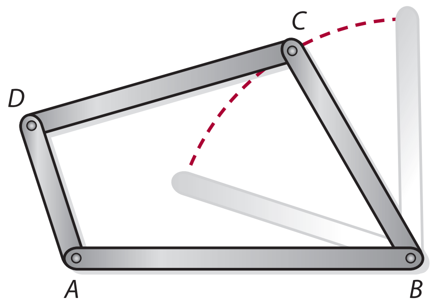
24 Explain why there is no Side-Side-Side-Side (SSSS) congruence condition for quadrilaterals.

25 Explain why opposite angles of a rhombus are congruent. Are both pairs of opposite angles of a kite congruent? Explain.

26 Look back at Problem 3 of Investigation 4. Could a builder also lay out a wall perpendicular to an existing wall by measuring the existing wall at 6 feet, the location of the new wall at 8 feet, and then check if the distance between the two points is 10 feet? Explain your reasoning. Which method would likely give greater accuracy? Why?

Extensions

27 The diagram at the right shows a quadrilateral linkage with frame $\overline{AB}$ satisfying Grashof's Principle that you discovered in Problem 6 of Investigation 1 (page 367). When the shortest crank $\overline{AD}$ makes a complete revolution, the other crank $\overline{BC}$ oscillates between two positions moving back and forth in an arc as indicated in the diagram.



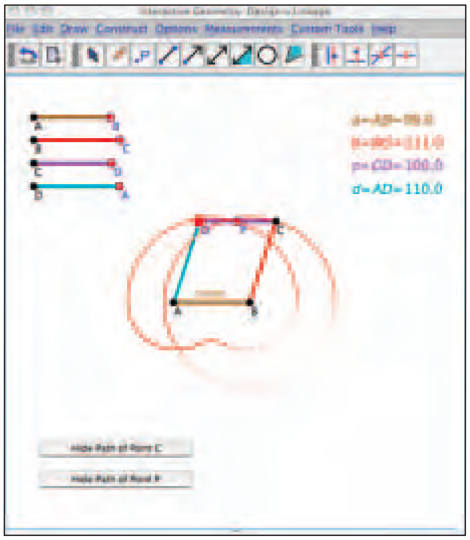
a. Use software like the "Design a Linkage" custom tool to investigate the possible paths of point C under the following two conditions.

i. Quadrilateral $ABCD$ is a kite.

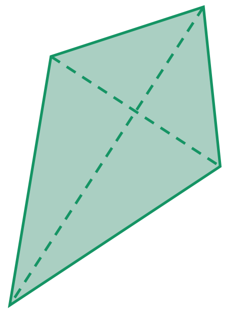
ii. Quadrilateral $ABCD$ is a parallelogram, including special types.

Consider the two cases where the frame is the longest or the shortest side. Write a paragraph summarizing your findings.

b. Repeat Part a for the case of a point $P \neq C$ on the coupler.



28 In order for kites to fly well, they need to have a high ratio of *lift area* to weight. For two-dimensional kites, the lift area is just the area of the kite.



a. On a copy of the kite shown, label its vertices and use markings to show which segments are congruent by definition of a kite.

b. Use congruent triangles to help you find the lift area of the traditional kite shown with cross pieces of lengths 0.8 m and 1.0 m.

c. Suppose the lengths of the diagonals of the kite are a and b where $a < b$. Use the diagram to help develop a formula for the area of a kite.

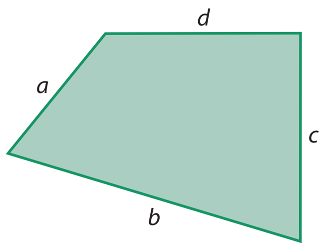
d. Can you also use your formula to find the area of a rhombus? Explain your reasoning.

e. Could this formula be used to find the area of any other quadrilaterals? Explain.

29 As noted at the beginning of this lesson, ancient Egyptians had to deal with changes in shape and size of fields caused by the annual flooding of the Nile River. Historians have evidence that the Egyptians calculated the areas of quadrilateral-shaped fields using the formula shown below.

$$A = \frac{1}{2}(a + c) \cdot \frac{1}{2}(b + d)$$

where a, b, c, d are the lengths of consecutive sides of the quadrilateral.



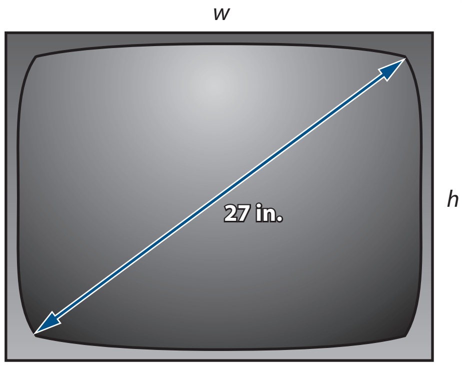
a. State this formula in words (without using the labels a, b, c, d) using the idea of "average."

b. Describe quadrilaterals for which the formula gives an exact calculation of the area.

c. Use software like the "Areas of Quadrilaterals" custom tool to explore cases of other quadrilaterals. For which quadrilateral shapes does the formula overestimate the area? Underestimate the area?

d. Explain why your findings in Part c make sense in terms of area formulas for parallelograms, trapezoids, and kites.

30 The television industry has set standards for the sizing of television screens. The ratio of height h to width w is called the *aspect ratio*. The aspect ratio for a conventional television screen is 3:4. That is $\frac{h}{w} = \frac{3}{4}$.



a. Write a rule expressing h as a function of w.

b. Use the Pythagorean Theorem to write a rule relating h, w, and the diagonal length 27.

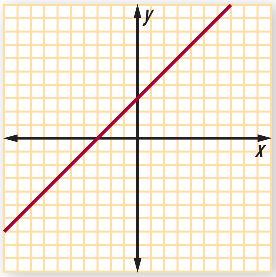
c. Use your rules in Parts a and b to find the standard dimensions of a 27-inch diagonal TV screen.

d. Check the dimensions you obtained against actual measurements of a 27-inch TV screen.

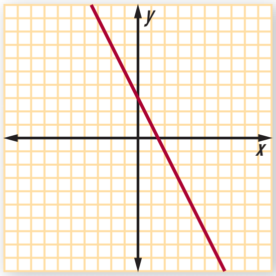
Review

31 Write an equation that matches each graph. The scale on both axes is 1.

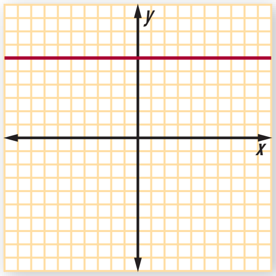
a.



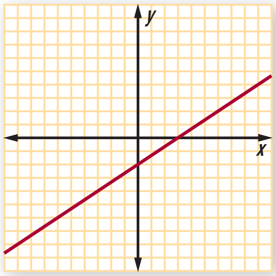
b.



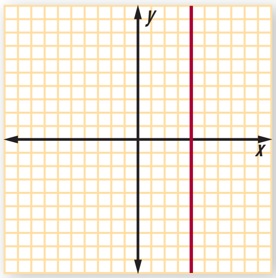
c.



d.



e.



On Your Own

32 You have a number cube with the numbers 2, 3, 4, 5, 6, and 7 on the faces. You roll the cube and look at the number showing on the top face.

a. What is the probability of rolling an even number?

b. What is the probability of rolling a prime number?

c. What is the probability of rolling a number less than 3?

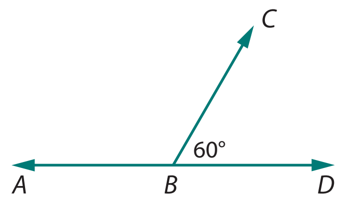
d. What is the probability of rolling an odd number that is greater than 4?

33 Use the fact that $36 \times 15 = 540$ and mental computation to evaluate the following.

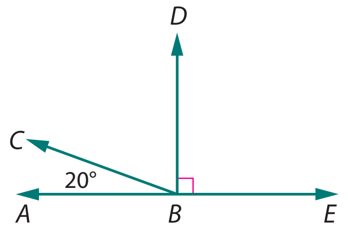
a. $\frac{5{,}400}{36}$ **b.** 3.6×15 **c.** 72×15 **d.** $\frac{45 \cdot 36}{3}$

34 Without measuring, find the measure of each indicated angle.

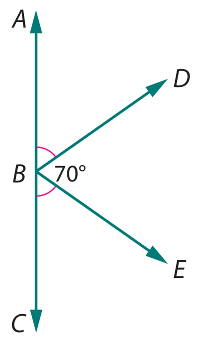
a. m∠*ABC*



b. m∠*CBD*



c. m∠*CBE*



35 Algebraic models can often help you analyze a situation. In this task, you will write algebraic rules for several different situations. Before you write each rule, think about whether a linear or an exponential rule would be better for the situation.

a. Alena's telephone credit card charges \$0.50 just to make a call and then charges \$0.04 for each minute.

i. Write a rule that gives the charge for a call based upon the length of the call in minutes.

ii. Alena wants to be able to make a 40-minute call to her friend. How much will this call cost her?

b. Randy owns a car that is currently worth \$8,750. The value of his car decreases by 15% each year.

i. Write a rule that gives the value of Randy's car t years from now.

ii. In how many years will Randy's car first be worth less than \$2,000?

36 Solve the following equations by reasoning with the symbols themselves.

a. $3x - 5 = 9x + 4$

b. $9 + \frac{1}{2}x = 14$

c. $3.2 = 5x + 0.7$

d. $2(4x - 8) = 8x + 14$

e. $2(5^x) = 250$

f. $(-2)(-2)^x = 16$

37 Find the value of each expression without using a calculator.

a. -5^2

b. $(-3)^2 - 4(2) + 21$

c. $148 - 3(-5)$

d. $\frac{-15 + 8(-3)}{2}$

e. $-6 + (3 - 5)^3$

38 Rewrite each expression in an equivalent form as an integer or radical expression in simplest form.

a. $\sqrt{\frac{16}{49}}$

b. $\sqrt{44}$

c. $\sqrt{3}\sqrt{15}$

d. $2\sqrt{63}$

e. $81^{\frac{1}{2}}$

f. $\frac{\sqrt{8}}{2}$

39 The table below gives the number of words spelled correctly (out of ten) by a group of students preparing for a spelling competition.

Number of Words Spelled Correctly	6	7	8	9	10
Number of Students	5	4	10	8	3

a. Calculate the mean and standard deviation of the number of words spelled correctly.

b. Colin and Lindsey tried to spell these ten words, and they both spelled all ten of the words correctly. Lindsey then added her score of 10, Colin's score of 10, and the average of the other 30 students and then divided that sum by 3 to get a new mean of 9.33. Is Lindsey's mean the correct mean of all 32 students? If not, explain the problem with her reasoning.

40 Often you will need to convert measures from one unit to another. Use what you know about seconds, minutes, hours, and days to complete each statement.

a. 80 minutes = _______ seconds = _________ hours

b. 3 days = _________ hours = _________ minutes

c. 300,000 seconds = ________ days = _________ years

LESSON 2

Polygons and Their Properties

Triangles and quadrilaterals are special classes of **polygons**—closed figures in a plane, formed by connecting line segments endpoint-to-endpoint with each segment meeting exactly two other segments. The segments are the *sides* of the polygons, and the points that they join are the *vertices*. Some other polygonal shapes that can be seen in daily life and with which you may be familiar are shown above.

Think About This Situation

As you examine the photos on the previous page, try to identify the polygon in each case and think about some of its features.

a How would you describe the shape of each polygon?

b What features do each of these polygons appear to have in common?

c The design of most bolts and nuts are based on polygons with an even number of sides. Why do you think this is the case? Why do you think the nuts on many public water mains and fire hydrants have the shape shown?

d Why do you think a stop sign has the shape it has? Would a square or rectangle work just as well?

e Why do you think the cells of a honeycomb are shaped as they are? Would other polygons work just as well?

f Based on your previous work with triangles and quadrilaterals, what are some natural questions you might ask about other polygons?

In this lesson, you will investigate properties of polygons, including relationships among their sides, angles, and diagonals. You also will explore the symmetry of polygons and patterns formed by combinations of polygons. These properties and patterns have important applications in art, design, and manufacturing.

Investigation 1 Patterns in Polygons

As the thin metal sheets that form the aperture of a camera move together or apart, they determine the amount of light that passes through a camera lens. The closing apertures on various cameras also determine polygons that differ in their number of sides.



Polygons can be classified in several different ways. One of the most commonly used classifications is in terms of the number of sides they have.

Number of Sides	Name	Number of Sides	Name
3	Triangle	9	Nonagon
4	Quadrilateral	10	Decagon
5	Pentagon	11	11-gon
6	Hexagon	12	Dodecagon
7	Septagon	15	15-gon
8	Octagon	*n*	*n*-gon

1 Name the polygons pictured on pages 398 and 399.

The polygonal shapes in the photos are examples of **regular polygons**. In regular polygons, all sides are congruent and all angles are congruent. These shapes also have a certain balance or regularity of form that can be explained in terms of their *symmetry*. As you work on the following problems, look for answers to these questions:

How can you accurately draw or build a regular polygon?

How can you describe the symmetry of a regular polygon and other shapes?

2 You can discover a method for accurately drawing a regular polygon by conducting the following experiment with a two-mirror kaleidoscope.

Hinge two mirrors together with tape so that you can adjust the angle between them. Draw a line segment on a sheet of paper and place a dot on the segment. Position the mirrors as shown in the photo. Adjust the mirrors so that they make an isosceles triangle with the segment and you can see a regular pentagon. Carefully trace the angle formed by the mirrors. Measure the angle.

a. Complete a table like the one below by adjusting the mirrors to form a regular polygon with the given number of sides.

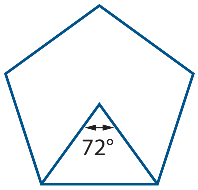
Regular Polygon	3	4	5	6	7	8
Angle Between Mirrors	?	?	72°	?	?	?

b. Predict the angle between the mirrors necessary to form a decagon. Check your prediction.

c. Write a rule that gives the measure M of the angle between the mirrors as a function of the number of sides of the regular n-gon produced by the two-mirror kaleidoscope.

The angle you traced in each case is called a **central angle** of the regular polygon.

d. To draw a regular pentagon, first draw a circle with a compass. Next draw a radius and then use a protractor to draw a central angle of 72°. The points where the sides of the angle intersect the circle are two of the vertices of the pentagon. How can you find the remaining three vertices using only a compass? Draw the pentagon.



e. Make an accurate drawing of a regular octagon.

f. Write a step-by-step description of a general method for drawing a regular n-gon.

3 If you rotate a tracing of a regular pentagon 72° about the center of the pentagon, the tracing will *coincide* (match) with the original figure. Try it. The regular pentagon has 72° **rotational symmetry**.

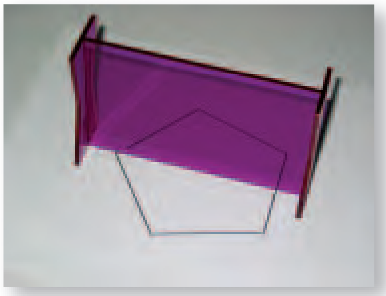
a. Explain why a regular pentagon also has 144° rotational symmetry. What other rotational symmetries does a regular pentagon have that are less than 360°? We do not consider a 360° rotation since a tracing of any figure will coincide with the original figure after a rotation of 360°.

b. What are the rotational symmetries of an equilateral triangle? Do other triangles have rotational symmetry? Explain.

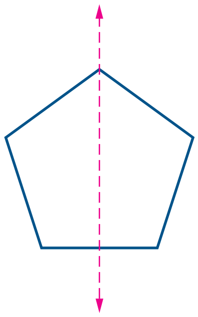
c. What are the rotational symmetries of a square? Of a regular hexagon?

d. Make a conjecture about the number of rotational symmetries of a regular n-gon. What angle measures will these symmetries have? Test your conjecture for the case of a regular octagon.

4 A regular pentagon also has **reflection** or **line symmetry**; sometimes called **mirror symmetry**.



When a mirror or piece of dark-colored Plexiglas is placed on a *line of symmetry*, half of the figure and its reflected image form the entire figure.



When a tracing of the figure is folded along the *line of symmetry*, one-half of the figure exactly coincides with the other half.

a. On a tracing of the regular pentagon above, draw each line of symmetry.

b. Draw an equilateral triangle and then draw each of its symmetry lines. What other triangles have lines of symmetry? What property or properties do these triangles have in common?

Jim Laser

c. Describe the lines of symmetry of a square.

d. Draw a regular hexagon and then draw its lines of symmetry.

e. Make and test a conjecture about the number of symmetry lines of a regular octagon.

f. Make a conjecture about the number of symmetry lines of a regular n-gon.

   i. Describe where the symmetry lines cut a regular n-gon when n is an even number.

   ii. Describe where the symmetry lines cut a regular n-gon when n is an odd number.

g. How is the line of symmetry of a figure related to the segment connecting a point on the figure with its mirror image?

5 Now consider the symmetries of special quadrilaterals that are not regular polygons—kites, general parallelograms, rhombuses, and rectangles.

a. Which of these other special quadrilaterals have line symmetry?

   i. In each case, sketch the shape and each of its symmetry lines.

   ii. Which of the quadrilaterals have symmetry lines that join vertices? What do these quadrilaterals have in common?

   iii. Which of the quadrilaterals have symmetry lines that do not join vertices? Where are the symmetry lines located? How do such quadrilaterals differ from those in part ii?

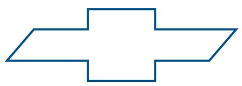
b. Which of these other special quadrilaterals have rotational symmetry?

   i. In each case, what are the angles of the rotational symmetries?

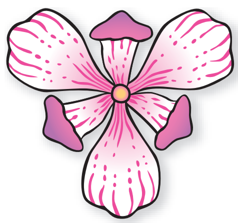
   ii. What property or properties do the quadrilaterals with rotational symmetry have in common?

6 Symmetry is perhaps the most common geometric characteristic of shapes found in nature. Symmetry is also often an integral part of the art forms created by people throughout history and across many cultures of the world. Examine carefully each of the figures shown below.

A


B


C


D


E


F


a. Which of these figures have reflection symmetry? Using a copy of these figures, for each figure with reflection symmetry, draw its line(s) of symmetry.

b. Which of the shapes have rotational symmetry? For each shape with rotational symmetry, give the angle(s) through which it can be rotated to coincide with itself.

c. If a figure has reflection symmetry, must it have rotational symmetry? Explain your reasoning.

d. If a figure has rotational symmetry, must it have reflection symmetry? Explain.

Summarize the Mathematics

In this investigation, you learned how to draw regular polygons and discovered special patterns relating the number of sides to the measure of a central angle and to the number and nature of rotational and reflection symmetries.

a. Explain how you would accurately draw a regular n-gon.

b. Explain how you can test to see if a figure has line symmetry. Describe the number and positions of the lines of symmetry of a regular n-gon.

c. Explain how you can test if a figure has rotational symmetry. Describe the number of rotational symmetries of a regular n-gon, and their angles.

Be prepared to share your ideas and thinking with the class.

Check Your Understanding

Make an accurate drawing of a regular nonagon.

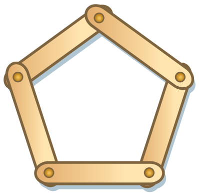
a. What is the measure of a central angle?

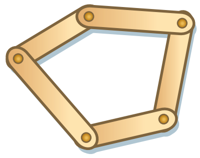
b. Describe all the rotational symmetries.

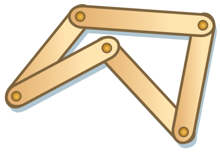
c. Sketch all the lines of symmetry. Where do the symmetry lines cut the sides of the nonagon?

Investigation 2 The Triangle Connection

Polygons with the same corresponding side lengths can have quite different shapes. As in the special case of quadrilaterals, polygons can be convex or nonconvex. In a **convex polygon**, no segment connecting two vertices is outside the polygon. Unless otherwise stated, in the remainder of this unit and in future units, polygons will be assumed to be convex.


Convex


Convex


Nonconvex

The shape and symmetry of a polygon depend on both side lengths and angle measures. As you work on Problems 1–4, look for answers to the following question:

How are the measures of the angles of any polygon related to the number of sides?

1 In Lesson 1, you learned that polygon shapes of four or more sides are not rigid. They can be made rigid by adding diagonal braces.

a. How could you use this idea of *triangulation* to find the sum of the measures of the interior angles of a pentagon? Compare your method and angle sum with others.

b. Use similar reasoning to find the sum of the measures of the interior angles of a hexagon. Why is it not necessary that the hexagon be a regular hexagon?

c. Complete a table like the one below for polygons having up to 9 sides. Examine your table for patterns relating sides, triangles, and angle sums.

Number of Sides	Number of Triangles	Sum of Interior Angles
4		
5	3	540°
6		
⋮		

d. Predict the sum of the measures of the interior angles of a decagon (10 sides). Check your prediction with a sketch.

e. Suppose a polygon has n sides. Write a rule that gives the sum of the measures of its interior angles S as a function of the number of its sides n.

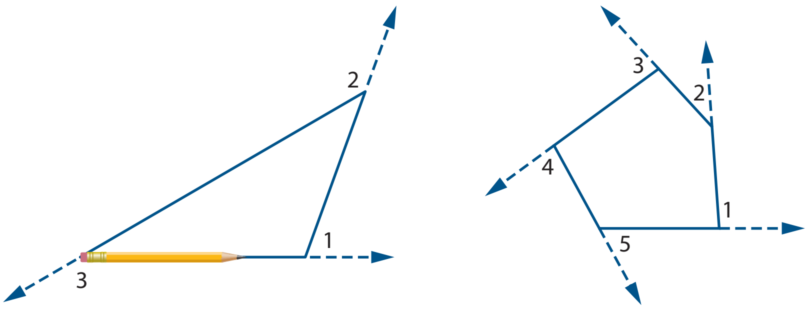
f. Test your rule for $n = 3$ (a triangle) and $n = 4$ (a quadrilateral).

g. Why is your function in Part e a linear function?

i. What is the slope of the graph of your function?

ii. What does the slope mean in terms of the variables? Does the y-intercept make sense? Why or why not?

2 By extending each side of a polygon, you create an *exterior angle* at each vertex.



a. Conduct the following exploration.

For each shape shown above:

(1) Place your pencil along the horizontal side of the shape with the tip pointing to the right as shown.

(2) Slide your pencil to the right until the eraser end is at the vertex of $\angle 1$.

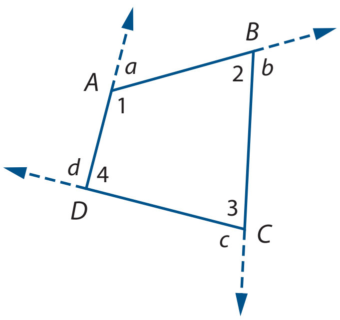
(3) Turn your pencil about the vertex so that it aligns with the second side.

(4) Repeat Steps 2 and 3 for each vertex until your pencil has made a complete trip around the polygon and is in its original position.

When the trip is completed, by how much has the pencil turned? What does this suggest about the sum of the measures of the exterior angles of the shape?

b. Repeat the exploration by drawing a different polygon of your choice. Did you find supporting evidence for your conjecture? What do you think is true about the sum of the measures of the exterior angles of any convex polygon?

3 Exterior angles can be created by extending sides in either direction. Suppose a, b, c, and d represent the measures of the exterior angles of quadrilateral $ABCD$.

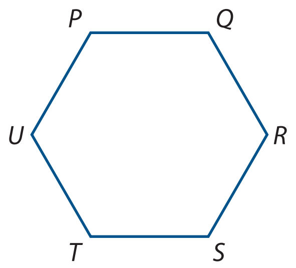


a. Use the diagram and careful reasoning to write a general argument for why your conjecture in Problem 2 is true for any quadrilateral. Compare your reasoning with others and resolve any differences.

b. Explain how you might use a similar argument in the case of an octagon.

4 If a polygon is a regular polygon, then you can find a relationship between the number of its sides and the measure of *each* interior angle.

a. What is the measure of an interior angle of a regular hexagon?



b. Write a rule that gives the measure A of an interior angle of a regular polygon as a function of its number of sides n.

c. What is the measure of an exterior angle of a regular hexagon? Of a regular 15-gon?

d. Write a rule that gives the measure E of an exterior angle of a regular polygon as a function of its number of sides n.

e. Visualize an example of a regular n-gon with an exterior angle drawn. If you added the expressions for your rules in Parts b and d, what should you get? Try it.

f. Are your functions in Parts b and d linear functions? If so, what is the constant rate of change in each case?

Summarize the Mathematics

In this investigation, you used reasoning with triangles to help discover a pattern relating the number of sides of a polygon to the sum of the measures of its interior angles. You also discovered a surprising pattern involving the sum of the measures of the exterior angles of a polygon.

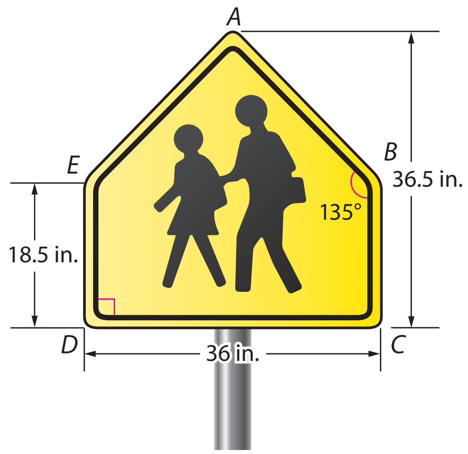
a Explain how you can find the sum of the measures of the interior angles of a polygon. What is the measure of one interior angle of a regular n-gon?

b What is true about the sum of the measures of the exterior angles of a polygon? What is the measure of one exterior angle of a regular n-gon?

Be prepared to share your ideas and reasoning with the class.

Check Your Understanding

Being able to recognize traffic signs by their shape and color is important when driving and is often tested on exams for a driver's license. Examine the school crossing sign at the right.



a. Identify the shape of the sign and describe the symmetries of this shape.

b. Use the design specifications shown and symmetry to find the lengths of the remaining sides of the sign.

c. Find the measures of the remaining interior angles.

d. On a copy of the shape, extend the sides to form an exterior angle at each vertex. Find the measure of each exterior angle.

Investigation 3 Patterns with Polygons

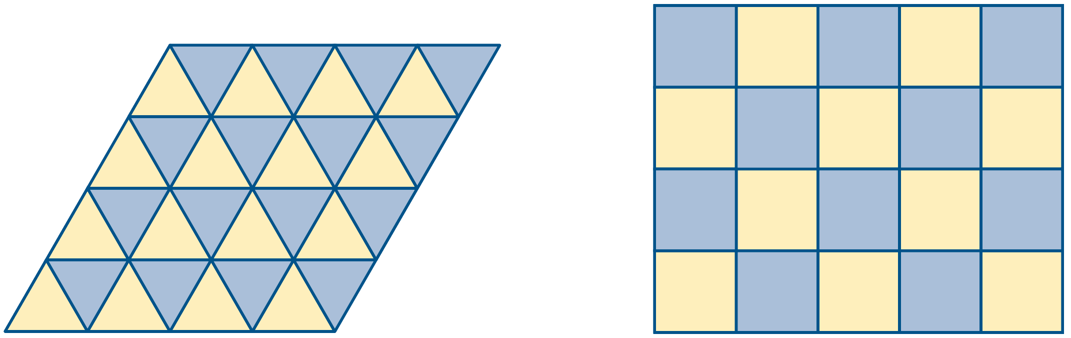
One of the most common and interesting applications of polygon shapes is their use as tiles for floors and walls. The photo below shows portions of two different *tilings* at the Center for Mathematics and Computing at Carleton College in Northfield, Minnesota. The portion of the tiling in the center of the photo is based on special tiles and a procedure for placing them created by Sir Roger Penrose, a British mathematician at the University of Oxford. Can you identify the types of polygons used for the Penrose tiles?

Later in this lesson you will have the opportunity to explore the variety of patterns that can be created with these tiles. Researchers have recently discovered that certain chemicals naturally organize themselves in similar patterns, some of which are used to make nonstick coating for pots and pans.

As you work on the following problems, make notes that will help you answer this question:

Which polygons or combinations of polygons will tile the plane?

1 The figures below show portions of **tilings** or **tessellations** of equilateral triangles and squares. The tilings are made of repeated copies of a shape placed edge-to-edge so that they completely cover a region without overlaps or gaps.



a. Assume that the tilings are extended indefinitely in all directions to cover the plane.

i. Describe the various ways that you can *slide* a tracing of each tiling so that it coincides with the original tiling. These tilings have **translation symmetry**.

ii. How could you describe the translation symmetries using arrows?

iii. Do the extended tilings have any reflection symmetry? If so, describe the lines of symmetry.

iv. Do the extended tilings have any rotational symmetries? If so, describe the centers and angles of rotation.

b. For these two tilings:

i. what is the sum of the measures of the angles at a common vertex?

ii. what is the measure of each angle at a common vertex?

c. In the tiling with equilateral triangles, identify other common polygons formed by two or more adjoined triangles that also produce a tiling. Sketch each and show the equilateral triangles that form the new tile. What does this suggest about other polygons that could be used to tile? Explain your reasoning.

2 Now explore if other triangles can be used as tiles.

a. Working in groups, each member should cut from poster board a small triangle that is *not* equilateral. Each member's triangle should have a different shape. Individually, explore whether a tiling of a plane can be made by using repeated tracings of your triangle. Draw and compare sketches of the tilings you made.

b. Can more than one tiling pattern be made by using copies of one triangle? If so, illustrate with sketches.

c. Do you think any triangle could be used to tile a plane? Explain your reasoning. You may find software like the "Tilings with Triangles or Quadrilaterals" custom tool helpful in exploring this question.

3 The most common tiling is by squares. In this problem, you will explore other quadrilaterals that can be used to make a tiling.

a. Each member of your group should cut a nonsquare quadrilateral from poster board. Again, each of the quadrilaterals should be shaped differently. Individually, investigate whether a tiling of a plane can be made with the different quadrilaterals. Draw sketches of the tilings you made.

b. Can more than one tiling pattern be made using the same quadrilateral shape? If so, illustrate and explain.

c. Make a conjecture about which quadrilaterals can be used to tile a plane.

i. Test your conjecture using software like the "Tilings wih Triangles or Quadrilaterals" custom tool.

ii. Explain why your conjecture makes sense in terms of what you know about angles of quadrilaterals.

4 You have seen two regular polygons that tile the plane. Now explore other regular polygons that could be used to make a tiling.

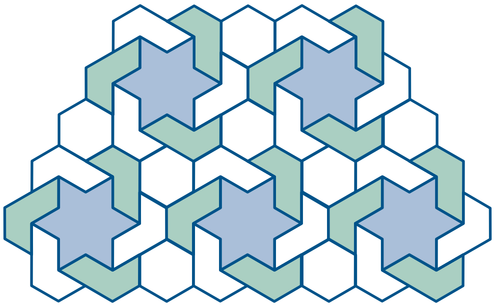
a. Can a regular pentagon tile the plane? Explain your reasoning.

b. Can a regular hexagon tile the plane? Explain.

c. Will any regular polygon of more than six sides tile the plane? Provide an argument to support your conjecture.

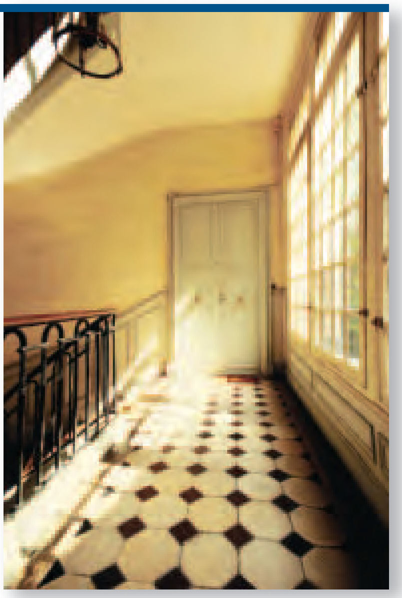
d. Tilings that consist of repeated copies of a single regular polygon with edges that match exactly are called **regular tessellations**. Which regular polygons can be used to make a regular tessellation?

5 As you saw at the beginning of this investigation, tilings can involve more than one type of shape. Another example of such a tiling is shown below. This tiling is from the Taj Mahal mausoleum in India.

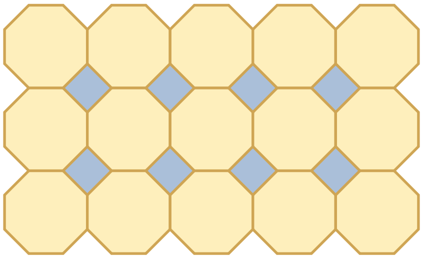


a. How many different shapes are used in the tiling? Draw a sketch of each shape.

b. Each shape can be divided into equilateral triangles, so this tiling is related to the tiling in Problem 1. On a copy of the tiling, show the equilateral triangles that make up each shape.

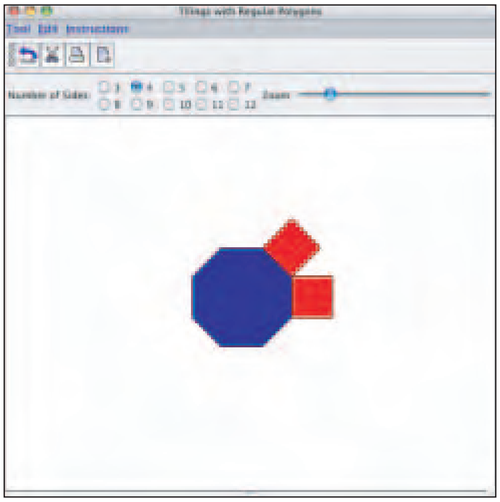


6 Combinations of regular octagons and squares are frequently used to tile hallways and kitchens of homes. They are also often used in the design of outdoor patios.



a. Explain why the polygons fit together with no overlaps or gaps. At each vertex, is there the same arrangement of polygons?

b. Tessellations of two or more regular polygons that have the same arrangement of polygons at each vertex are called **semiregular tessellations**. Use tiles made from poster board or software like the "Tilings with Regular Polygons" custom tool to test whether there can be a semiregular tessellation that has at each vertex a regular hexagon, two squares, and an equilateral triangle. If possible, draw a sketch of such a tessellation.



c. Semiregular tessellations are coded by listing the number of sides of the polygons that meet at a single vertex. The numbers are arranged in order with a smallest number first. The tessellation above Part a is coded 4, 8, 8, for square, octagon, octagon.

i. Use this code to describe the tessellation you drew in Part b. Give the code for each of the three possible regular tessellations.

ii. Determine if the vertex arrangement 3, 6, 3, 6 describes a semiregular tessellation.

iii. Is there another semiregular tessellation that can be formed using equilateral triangles and regular hexagons? Explain.

Mel Curtis/Getty Images

d. There are eight different semiregular tessellations. You have examples of four of them. Find at least one more example and name it using the vertex arrangement code in Part c. Compare your findings with those of other classmates.

Summarize the Mathematics

In this investigation, you explored special polygons that tile the plane. You also investigated how combinations of those special polygons can lead to more complex patterns in the plane.

a Write a summarizing statement describing which triangles and which quadrilaterals tile the plane.

b Which regular polygons tile the plane? How do you know there are no others? Explain your reasoning.

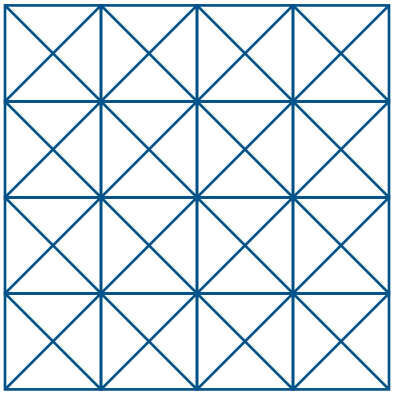
c How do semiregular tessellations differ from regular tessellations? How can number codes be used to describe semiregular tessellations?

Be prepared to discuss your ideas with the class.

Check Your Understanding

You have seen that a regular pentagon will not tessellate the plane. There are nonregular convex pentagons that will tessellate. But not many. Researchers have identified only 14 types. Whether there are more remains an open question.

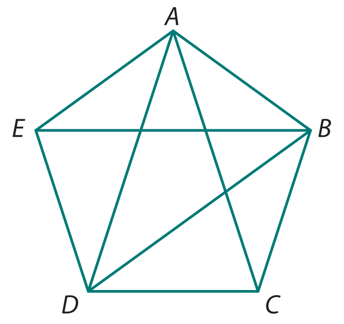
Using a copy of the figure at the right, find a pentagon in the figure that will tile the plane. Shade it. Show as many different tiling patterns for your pentagon as you can.



On Your Own

Applications

1 In regular pentagon *ABCDE*, three diagonals have been drawn.



a. Use careful reasoning to explain why $\overline{AD} \cong \overline{AC}$.

b. Is diagonal $\overline{AD}$ congruent to diagonal $\overline{DB}$? Explain your reasoning.

c. Give an argument for why diagonal $\overline{AD}$ is congruent to diagonal $\overline{EB}$.

d. Are all the diagonals of a regular pentagon congruent? Explain your reasoning.

2 Suppose every student in your mathematics class shook hands with each of the other students at the beginning of your next class. What would be the total number of handshakes?

a. Show how you can represent this problem with a polygon for the case of just 4 students.

i. How many handshakes were involved?

ii. How is your answer related to the components of the polygon?

b. Represent and solve the problem for the special case of 5 students. Of 6 students.

c. How did you represent students in your models? How did you represent handshakes between students?

d. Use any numerical or visual pattern in your models to help solve the original problem.

e. In addition to using patterns to solve problems, it is also important to be able to explain and, whenever possible, generalize the patterns you discover. A student in a Wisconsin classroom claimed that a class of n students would involve $n^2 - 2n$ handshakes. The student reasoned as follows.

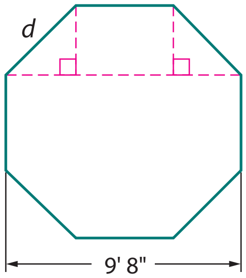
> I thought of students as vertices of an n-gon and handshakes as the sides and diagonals. An n-gon has n sides and n angles. From each vertex, I can draw a diagonal to $n - 3$ other vertices. So, I can draw $n(n - 3)$ diagonals. So, the number of handshakes is $n + n(n - 3) = n^2 - 2n$.

i. Is this reasoning correct? If not, identify and correct errors in the reasoning.

ii. Write a rule that expresses the number of handshakes as a function of the number of students n.

f. Write in words a general rule for calculating the number of diagonals of an n-gon.

3 In designing this mall garden, the architect proposed a gazebo in the shape of a regular octagon. The octagonal roof cupola was to have a width of 9 feet 8 inches. What should be the dimensions of each side? (*Hint*: Use the *auxiliary lines* drawn to help guide your thinking.)



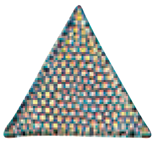
4 Objects in nature are often approximately symmetric in form.

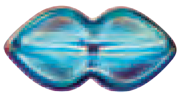
a. The shapes below are single-celled sea plants called *diatoms*.

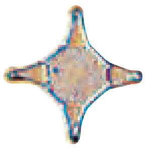
i. Identify all of the symmetries of these diatoms. Ignore interior details.

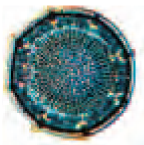
ii. For those with reflection symmetry, sketch the shape and show the lines of symmetry.

iii. For those with rotational symmetry, describe the angles of rotation.

A 

B 

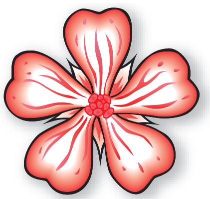
C 

D 

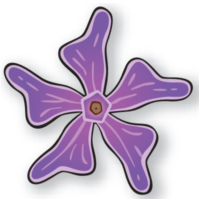
b. Identify all of the symmetries of the two flowers shown below.

i. If the flower has line symmetry, sketch the shape and draw all lines of symmetry.

ii. If the flower has rotational symmetry, describe the angles of rotation.

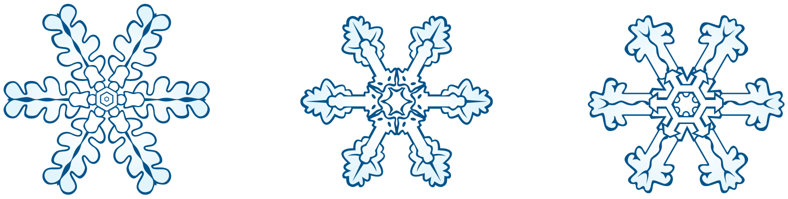
A 

Geranium

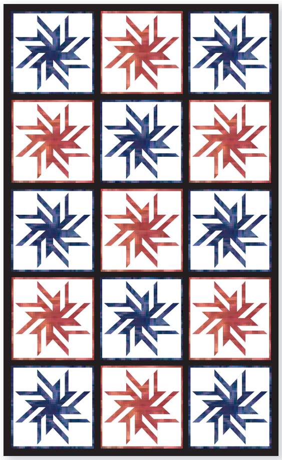
B 

Periwinkle

c. It has been said that no two snowflakes are identical. Yet every snowflake has some common geometric properties.
   i. Identify the symmetries of the snowflakes below.
   ii. In terms of their symmetry, how are the snowflakes alike?



Native American rug



Star of Bethlehem quilt

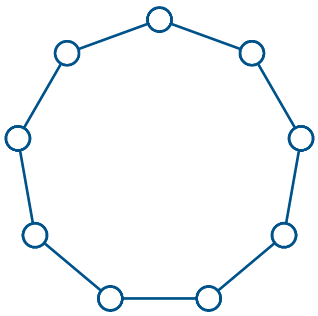
5. Polygons and symmetry are important components of the arts and crafts of many cultures.

   a. Examine the photo of a Native American rug.
      i. Describe the line and rotational symmetries of this rug. Sketch two design elements within the rug that have rotational symmetry. Describe the angles through which each can be turned.
      ii. Sketch two design elements within the rug that have line symmetry. Draw the line(s) of symmetry.
      iii. Are there any design elements which have both rotational and line symmetry? If so, identify them. Where is the center of rotation in relation to the lines of symmetry?

   b. The design of the quilt to the left is called "Star of Bethlehem."
      i. What rotational symmetries do you see in the fundamental "pinwheel stars"? Give the angles of rotation for each of these symmetries.
      ii. Is there line symmetry in the "pinwheel stars"? Explain.
      iii. Does the quilt as a whole, including the pinwheel stars, have rotational or line symmetry? Describe each symmetry you find.

6. Here is a two-person game that can be played on any regular polygon. To play, place a penny on each vertex of the polygon. Take turns removing one or two pennies from adjacent vertices. The player who picks up the last coin is the winner.

   a. Suppose the game is played on a nonagon, as shown at the right. Try to find a strategy using symmetry that will permit the second player to win always. Write a description of your strategy.



   b. Will the strategy you found work if the game is played on any polygon with an odd number of vertices? Explain your reasoning.

   c. Suppose the game is played on a polygon with an even number of vertices, say an octagon. Try to find a strategy that will guarantee that the second player still can win always. Write a description of this strategy.

7 Bees produce honeycomb cells with cross sections that are approximately the shape of a regular hexagon.

a. If all the cells are to be congruent, what other regular polygon shapes might they have used?

b. Suppose the perimeter of a cross section of one cell of a honeycomb is 24 mm. Find the area of the cross section, assuming the cell has the following shape.

  i. an equilateral triangle

  ii. a square

  iii. a regular hexagon

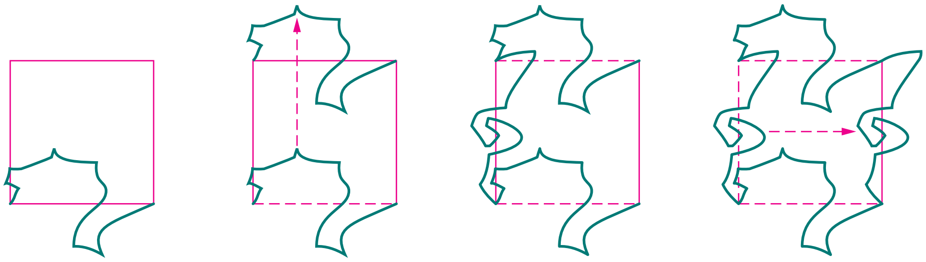
c. Which cell has the greatest cross-sectional area for a fixed perimeter of 24 mm? As the number of sides of a regular polygon with fixed perimeter increases, how does the corresponding area change?

d. Write a statement summarizing how shape is an important factor in the design of the cells of a honeycomb.

8 It is possible to create intriguing tessellations by carefully modifying the sides of a polygon. The Dutch artist M.C. Escher was a master of these modifications. He created this tessellation of *Pegasus*, the mythical winged horse.

a. Assuming the tessellation above is extended indefinitely in all directions, describe its symmetries.

b. Study the process below which illustrates how Escher may have created his "flying horse" tessellation from a square.



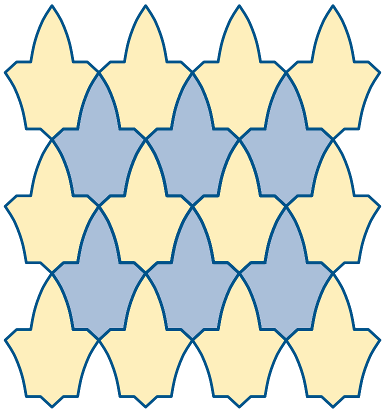
Step 1 Step 2 Step 3 Step 4

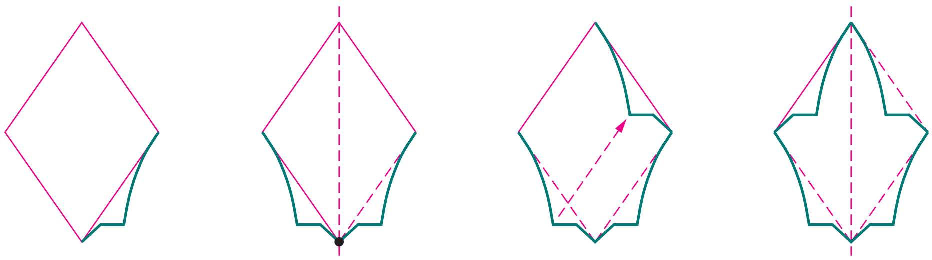
i. Use tracing paper to verify how the modifications of sides are translated to the opposite sides.

ii. How does the area of the Pegasus compare to the area of the initial square?

c. Start with a square, rectangle, or other parallelogram and use a similar process to create your own Escher-type tessellation. Verify that your shape does tile the plane.

9 A beautiful tiling for paving or wallpaper can be made with leaf-like tiles based on a rhombus. Study the process below, which illustrates how this tile may be created.





Step 1 Step 2 Step 3 Step 4

a. Use tracing paper to verify how the modifications of sides of the rhombus are reflected or translated to produce the other sides of the leaf tile.

b. How does the area of the leaf tile compare to the area of the rhombus?

c. Start with a rhombus and use a similar process to create your own paving or wallpaper design.

Connections

10 Psychologists often use figures like that below in their study of human perception.

a. What do you see?

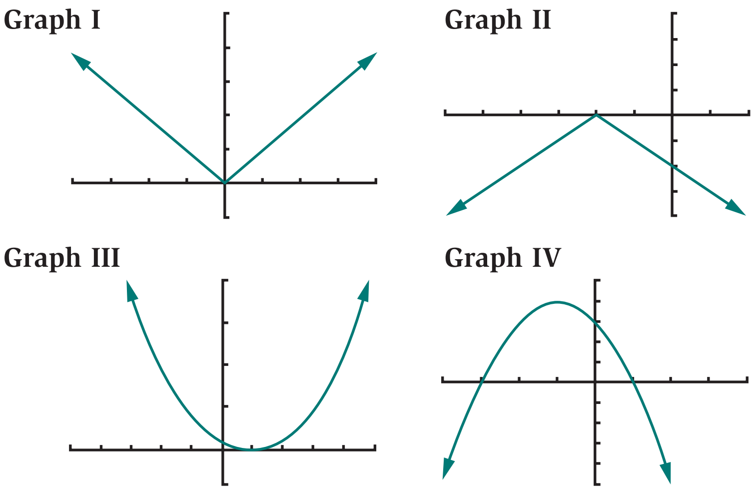
b. Name the colored polygons involved in this task.

c. Which polygons are convex? Nonconvex?

d. Which polygons have symmetry? Describe the symmetries.

11 Graphs of various functions relating variables x and y are shown below. The scale on the axes is 1.

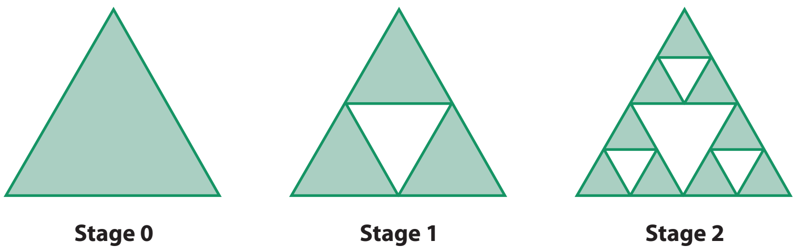
a. For each graph, locate any line(s) of symmetry.



b. What pattern do you see in the coordinates of the points on each line of symmetry? Write the equation of the symmetry line(s).

c. Suppose you have a graph, and its line of symmetry is the y-axis. If one point on the graph has coordinates $(-8, -23)$, what is the y-coordinate of the point on the graph with x-coordinate 8? Explain your reasoning.

12 The initial and first two stages in making a triangular Sierpinski carpet are shown below. In the *Exponential Functions* unit, you explored the area of carpet remaining as a function of the cutout stage. Describe the symmetries that the triangular carpet has at each stage of the process.



On Your Own

13 Make a sketch of a regular pentagon. Extend one of the sides to form an exterior angle. How could you use your knowledge of exterior angles to calculate the measure of an interior angle of a regular pentagon? Would your method work for other regular polygons? Explain your reasoning.

14 In Investigation 2, you discovered the rule $A = \frac{(n-2)180°}{n}$, which gives the measure A one interior angle of a regular n-gon.

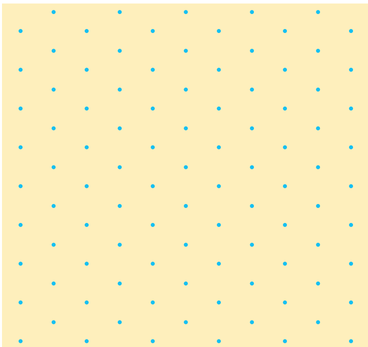
a. As the number of sides of a regular polygon increases, how does the measure of each of its interior angles change? Is the rate of change constant? Explain.

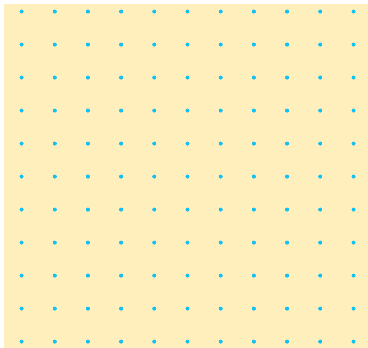
b. Use your formula to find the measure of one interior angle of a regular 20-gon. Could a tessellation be made of regular 20-gons? Explain your reasoning.

c. When will the measure of each angle of a regular polygon be a whole number?

d. Use your calculator or computer software to produce a table of values for angle measures of various regular polygons. Use your table to help explain why the only regular tiling of the plane is one with regular polygons of 3, 4, or 6 sides.

15 The **dual** of a tessellation by regular polygons is a new tessellation obtained by connecting the centers of polygons that share a common edge. Use equilateral triangular (isometric) dot paper to complete Parts a and b and square dot paper to complete Part c.



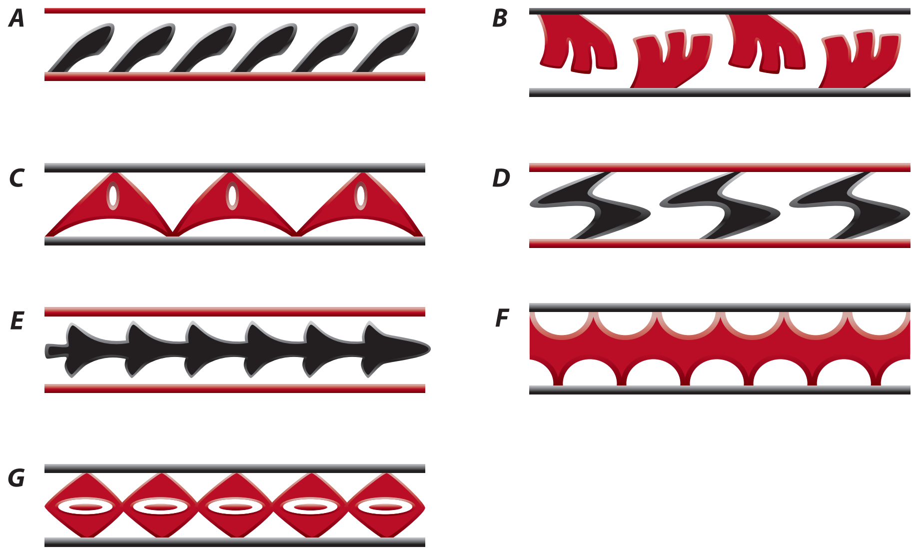


a. Draw a portion of a regular hexagon tessellation. Using a different colored pencil, draw and describe the dual of the tessellation.

b. Draw a portion of an equilateral triangle tessellation. Draw and describe the dual of this tessellation.

c. Draw and describe the dual of a tessellation of squares.

16 Strip or *frieze* patterns are used in architecture and interior design. The portions of frieze patterns below came from the artwork on pottery of the San Ildefonso Pueblo, New Mexico.



Source: Groups and Geometry in the Ceramic Art of San Ildefonso. *Algebras, Groups and Geometries 2*, no. 3 (September 1985).

Imagine that each frieze pattern extends indefinitely to the right and to the left.

a. Confirm that each pattern has translation symmetry.

b. Examine each frieze pattern for reflection and rotational symmetries. (If a strip has such a symmetry, the strip must appear the same before and after it is reflected across a line or rotated about a point.) Describe your findings for each pattern.

c. Pattern B has a symmetry called *glide-reflection symmetry*. Describe the translation and reflection combination that will move the motif so the pattern coincides with itself.

Reflections

17 Look up the word “polygon” in a dictionary. What is the meaning of its parts “poly” and “gon”? How do these meanings relate to your understanding of “polygon”?

18 A regular polygon is both equilateral and equiangular.

a. Give an example of an equilateral polygon that is not a regular polygon.

b. Give an example of an equiangular polygon that is not a regular polygon.

19 Recall that an Euler circuit is a route through a connected vertex-edge graph such that each edge of the graph is traced exactly once, and the route starts and ends at the same vertex. How are polygons similar to and different from Euler circuits in graphs?

20 Thumb through the yellow pages of a phone directory or visit Internet sites to find company logos. Why do you think so many of the logo designs are symmetric? Draw three of the logos that have particularly interesting symmetries.

21 Cross-cultural studies suggest that symmetry is a fundamental idea that all people use to help understand, remember, compare, and reproduce forms. However, symmetry preferences have been found across cultures. One study found that symmetry about a vertical line was easier to recognize than symmetry about a horizontal line. The study also found that symmetry about a diagonal line was the most difficult to detect. (Source: Orientation and symmetry: effects of multiple, rotational, and near symmetries. *Journal of Experimental Psychology* 4[4]: 1978.)

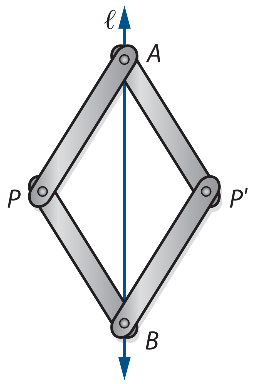
a. Would the findings of the study apply to the way in which you perceive line symmetry?

b. Describe a simple experiment that you could conduct to test these findings.

22 Look back at your work on Applications Task 1. Do you think your finding applies to other regular polygons? Carefully draw a regular hexagon and label its vertices *PQRSTU*. Name the diagonals that are congruent to each other and give reasons to support your conclusions.

Extensions

23 Commercial artists sometimes use a device similar to the one shown for drawing mirror images. The linkage is assembled so that $APBP'$ is a rhombus and the points at P and P' pivot freely as A and B slide along ℓ. As P traces out a figure, P' traces out its reflection image. Explain why this device works as it does.

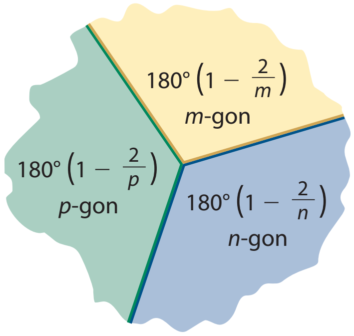


24 Use interactive geometry software or other tools to further investigate properties of regular polygons.

a. Is there a relationship between the measure of a central angle and the measure of an interior angle? If so, can you explain why that must *always* be the case?

b. Is there a relationship between the measure of a central angle and the measure of an exterior angle? If so, can you explain why?

25 In Investigation 3, you explored semiregular tessellations by examining various arrangements of regular polygon shapes around a vertex. You can also examine the possibilities using algebra. The diagram below is a start. It shows the case of three regular polygons of m, n, and p sides completely surrounding a vertex with no overlapping.



a. Why must the numbers m, n, and p all be integers greater than 2?

b. Explain why the measure of each interior angle is as shown.

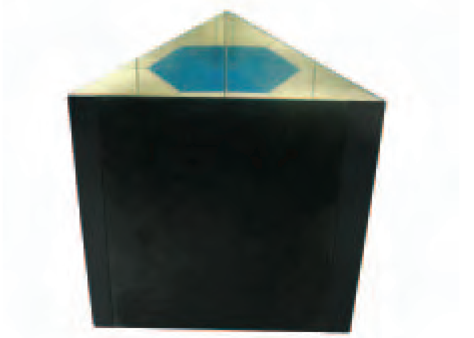
c. Write an equation that must be satisfied if the polygons are to form a tessellation.

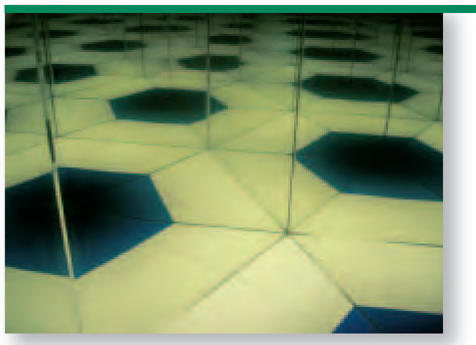
d. Show that your equation is equivalent to $\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = \frac{1}{2}$. What are the whole number solutions to this equation?

e. Relate one of your solutions in Part d to a semiregular tessellation.

26 In Investigation 1, you explored how a two-mirror kaleidoscope could be used to create regular polygons. For this task, make a three-mirror kaleidoscope by fastening three congruent mirrors together to form an equilateral triangle at the base.

a. Explore what can be created by placing various patterns in the base like those shown below.



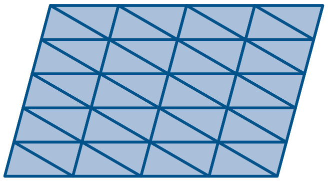


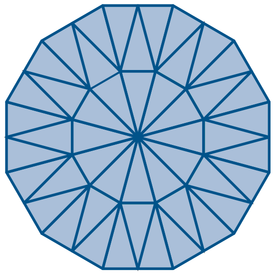
b. Can you create each regular tessellation by placing an appropriate pattern in the base of your kaleidoscope? If so, draw diagrams of the patterns.

c. Can you create a semiregular tessellation by placing an appropriate pattern in the base? If so, draw a diagram of the pattern.

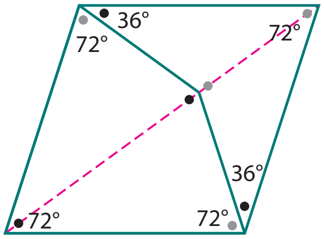
d. How are ideas of symmetry related to your pattern-building explorations?

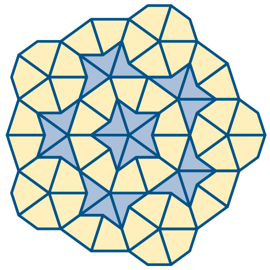
27 The first two tessellations on page 422 use the same isosceles triangle. Think of each tiling extended indefinitely to cover the plane. A tessellation that fits exactly on itself when translated is called **periodic**; it has translation symmetry. One that does not have translation symmetry is called **nonperiodic**. Many polygonal shapes (like the triangles below) will tile both periodically and nonperiodically. Many shapes (such as Escher's *Pegasus*) will tile *only* periodically.





The diagram below at the left shows how the Penrose tiles at the beginning of Investigation 3 are formed from a particular rhombus. If the vertices are color-coded with black dots and gray dots as shown and only vertices with the same color dots are allowed to meet, then the only way these kite and dart shapes can tessellate is nonperiodically.

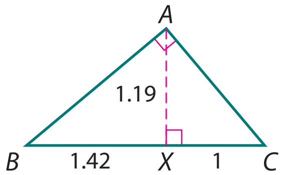




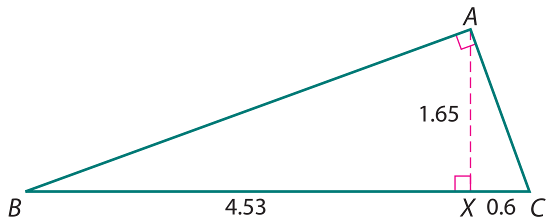
Use the "Tilings with Penrose Tiles" custom tool or tracing paper to make a copy of the kite and dart shown above. Then create two different nonperiodic tilings using the matching rule and explain why they are different.

Review

28 There are three right triangles in each of the following diagrams. In each case, you are given three segment lengths.



I



II

a. Using the marked segment lengths, find AB and AC in each case.

b. Find the ratio AB:AC and AX:XC in each case.

29 Russell's department store is having a 25%-off sale. There is 6% sales tax on all items bought.

a. Liz found a dress she likes that usually costs \$65. She has \$50. Does she have enough money to buy the dress during this sale?

b. Jerry bought a shirt on sale and paid a total of \$22.26. What is the price of the shirt when it is not on sale?

c. Write a formula that describes the relationship between the original price of an item p and the amount of money that is needed to buy the item during this sale C.

d. Is the relationship described in Part c linear, exponential, or neither? Explain your reasoning.

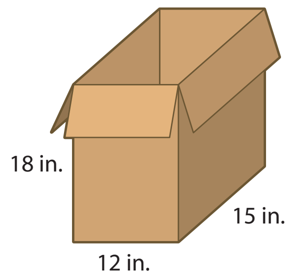
30 Consider the rule $2x - 4 = y$.

a. How do the y values change as the x values increase by 1?

b. How do the y values change as the x values increase by 5?

c. Where does the graph of this rule intersect the y-axis?

31 Janelle creates and sells small ceramic figurines at arts and crafts fairs. How many figurine boxes, which are 3 inches long, 2 inches wide, and 3 inches tall, could she fit into the shipping box shown at the right?



32 Write each of the following in a simpler equivalent exponential form that uses only positive exponents.

a. $(4x^3)^2$ **b.** $(5x^2y)(-3x^5y)$ **c.** $\frac{24t^6r^2}{8t^3r^5}$

d. $\left(\frac{-6d^4}{2}\right)^3$ **e.** $\left(\frac{5x^2y}{7x^5}\right)\left(\frac{-7x}{3y}\right)^2$

33 Write each expression in simplest equivalent form. Show your work.

a. $7(4 - x) + 15$ **b.** $36 - (50 - 15x) + 10x$

c. $0.5x(4 - 20) + \frac{6x + 4}{2}$ **d.** $7 + 4(5x - 7) - 2(15 - 9x)$

34 Complete each table so that its entries show an exponential pattern of change. Then write a rule that expresses y as a function of x.

a.

x	0	1	2	3	4
y	800	200	50		

b.

x	1	2	3	4	5
y	30	45	67.5		

c.

x	0	2	4	6	8
y	1	4	16		

35 Solve each equation.

a. $48 = 3(2^x)$ **b.** $129 = 6 + 3x$

c. $900(0.4^x) = 90$ **d.** $5(2x + 4) = 300$

36 Mariah wants to join a gym. In her community she has two choices. The First Street Gym charges a membership fee of \$120 and a monthly fee of \$35. Fitness Center has a \$20 membership fee and charges \$55 per month.

a. For what number of months will the total cost be the same at the two gyms?

b. Under what conditions will First Street Gym be less expensive?

LESSON 3

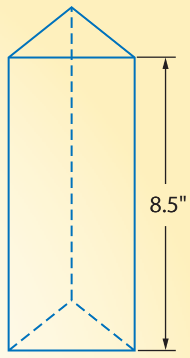
Three-Dimensional Shapes

In the previous two lessons, you studied two-dimensional shapes and some of their important properties and applications in building, design, and art. In many cases, two-dimensional shapes and their properties have corresponding ideas in three dimensions. In three dimensions, as in two dimensions, the shape of an object helps to determine its possible uses.

In your history classes, you may have noticed that the Greeks and people in other ancient cultures often used columns in the design of their buildings. The Greek Parthenon shown above is made of marble. Thus, the columns had to support great weights. An important design consideration is the shape of the column.

Think About This Situation

Suppose you made three columns by folding and taping sheets of 8.5 × 11-inch paper so that each column was 8.5 inches high with bases shaped, respectively, as equilateral triangles (pictured at the right), squares, and regular octagons. Imagine that you placed a small rectangular piece of cardboard (about 6 by 8 inches) on top of a column. Then you carefully placed a sequence of objects (like notebooks or textbooks) on the platform until the column collapsed.



a Which column shape do you think would collapse under the least weight?

b Which column shape do you think would hold the most weight before collapsing?

c Test your conjectures by conducting the experiment with your class. What happens to the maximum weight supported as the number of sides of the column base increases?

d Suppose another column is made with a regular hexagon base. Predict how the number of objects it would support before collapsing would compare to those of the three columns above. Explain your reasoning.

e Why do you think the ancient Greeks chose to use cylindrical columns?

In this lesson, you will use properties of two-dimensional shapes to aid in examining three-dimensional shapes. You will learn how to identify and describe common three-dimensional shapes and how to construct three-dimensional models of them. You will develop skill in visualizing and sketching them in two dimensions and in identifying their symmetries and other important properties. You will also explore some of the many connections between two- and three-dimensional shapes.

Investigation 1 Recognizing and Constructing Three-Dimensional Shapes

Like the columns of the Parthenon, most everyday three-dimensional shapes are designed with special characteristics in mind. In this investigation, you will consider the following questions:

What are important characteristics of common three-dimensional shapes?

How can three-dimensional models of these shapes be constructed?

1 As a class, examine the objects depicted below.

Popcorn Box **Candy Package** **Block "L"** **Ice Cream Cone**

a. Which of these objects have similar geometric characteristics? What are those characteristics?

b. How would you describe the shapes of the above objects?

c. A **polyhedron** (plural: *polyhedra*) is the three-dimensional counterpart of a polygon. It is made up of a set of polygons that encloses a single region of space. Exactly two polygons (*faces*) meet at each *edge* and three or more edges meet at a *vertex*. Furthermore, the vertices and edges of the polyhedron are vertices and edges of the polygon faces.

- **i.** Which of the above shapes are polyhedra? For each polyhedron, name the polygons that are its faces.
- **ii.** If a shape is not a polyhedron, explain why not.

d. A **convex polyhedron** is a polyhedron in which no segment connecting any two vertices goes outside the polyhedron. Which of the above shapes are convex polyhedra?

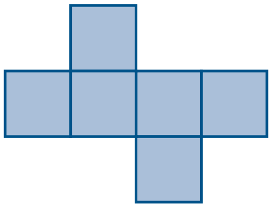
e. Name at least two other common objects with different polyhedron shapes.

2 Polyhedron-shaped boxes for packaging products like cereal and candy are often manufactured using a two-dimensional pattern, called a **net**, that can be folded along its edges to form the polyhedron.

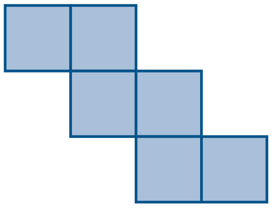
Matt Meadows

a. A **cube** is a polyhedron with six congruent square faces. Examine enlarged copies of the three nets shown below. Which of these nets can be folded to make a cube? For nets that can be folded to make a cube, make matching tick marks on edges that match when the net is folded into a cube.

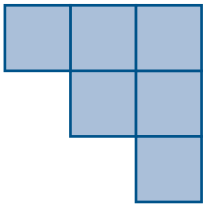
i.



ii.



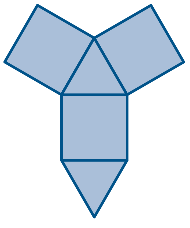
iii.



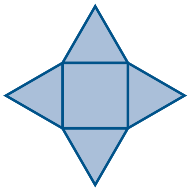
b. Draw a possible net for the candy package shown in Problem 1.

c. Using enlarged copies of the three nets pictured below, fold each net into a polyhedron. Divide the work among your classmates so that each student makes one polyhedron.

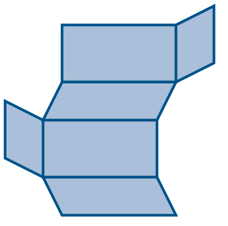
i.



ii.



iii.

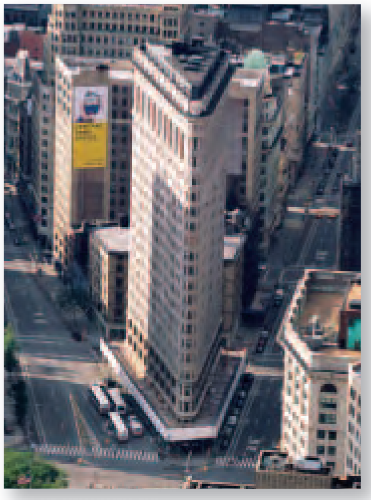


d. Examine each vertex of the three polyhedra you made in Part c. What is the least number of faces that meet at a vertex? What is the least number of edges that meet at a vertex? Would it be possible for fewer faces or edges to meet at a vertex of any polyhedron? Explain your reasoning.

e. When polygons tile the plane, the sum of the measures of angles at a vertex is 360°.

- **i.** What is the sum of the measures of angles that meet at each vertex of the first two polyhedra in Part c?
- **ii.** What is the sum of the measures of the angles that meet at each vertex of the cube in Part a? Of the block "L" in Problem 1?
- **iii.** Make a conjecture about the sum of the measures of angles that meet at each vertex of a convex polyhedron.

3 Two important types of polyhedra are *prisms* and *pyramids*. Architects frequently use these shapes and an understanding of their characteristics in the design of buildings. For example, the Flat Iron Building, designed by David Burnham and the oldest surviving skyscraper in New York City, has an unusual prism shape. The glass pyramid, pictured on the right, was designed by I. M. Pei for the entrance to the Louvre museum in Paris, France.



a. A **prism** is a polyhedron with two parallel congruent faces with corresponding edges that are connected by parallelograms (called *lateral faces*). Which of the four polyhedra in Problem 2 are prisms?

b. Either one of a pair of congruent, parallel faces of a prism may be called a *base of the prism*. How many different faces of each of the prisms that you identified in Part a could be considered to be a base of the prism? Explain your answers.

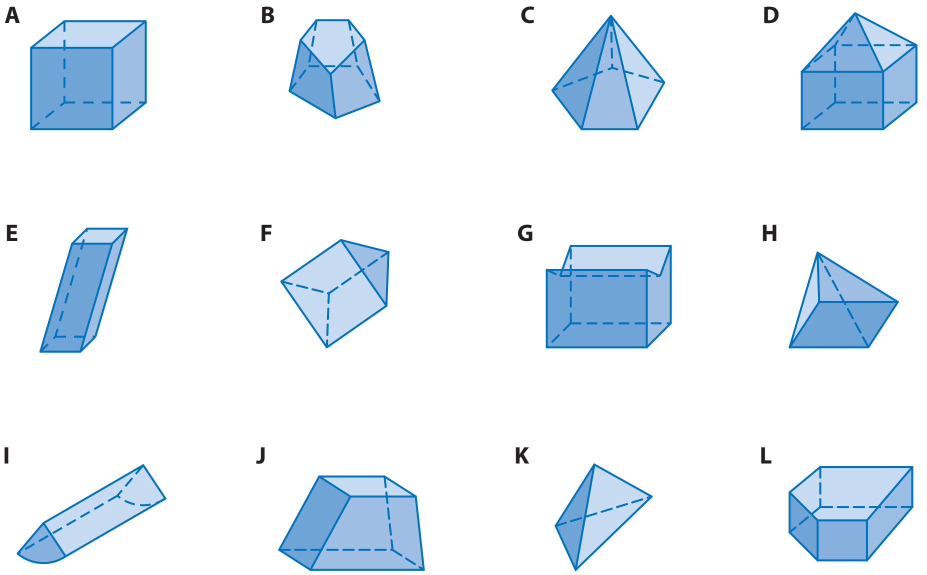
c. Compare the prisms in Problem 2 Part c. One prism is a *right prism* and the other is an *oblique prism*. From what you know about other uses of the term "right" in mathematics, which prism is a right prism? Write a definition of a right prism.

d. A **pyramid** is a polyhedron in which all but one of the faces must be triangular, and the triangular faces share a common vertex called the *apex of the pyramid*. The triangular faces are called *lateral faces*. The face that does not contain the apex may have any polygonal shape, and this face is called the *base of the pyramid*. Identify the base and apex of the pyramid in Problem 2 Part c.

e. Prisms and pyramids are often named by the shapes of their bases. For example, a pyramid with a five-sided base is called a *pentagonal pyramid*. Use this naming method to name a cube as a prism. Name the other prisms and pyramids in Problem 2.

4 Now examine each of the shapes below.

a. Which shapes appear to be right prisms? Oblique prisms? Pyramids? None of these? Explain your choices based on the definitions in Problem 3.

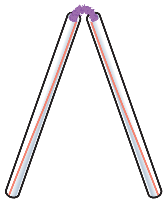


b. Which of the above polyhedra appear to be nonconvex? Explain.

Unless otherwise stated, in the remainder of this unit and in future units, prisms will be assumed to be convex right prisms.

5 Models for polyhedra can be hollow like a folded-up net or an empty box, or they can be solid like a candy bar or brick. Polyhedra can also be modeled as a skeleton that includes only the edges, as in a jungle gym.

Working in a group, make the following models from coffee stirrers or straws and pipe cleaners. Cut the pipe cleaners into short lengths that can be used to connect two straw edges at a vertex, as shown. Divide the work. Each student should build at least one prism and one pyramid. Save the models to explore properties of polyhedra later in this lesson.



Pacific Domes

a. cube: 5-cm edges

b. triangular prism: 5-cm edges on bases, 8-cm height

c. square prism: 5-cm edges on bases, 8-cm height

d. pentagonal prism: 5-cm edges on bases, 8-cm height

e. hexagonal prism: 5-cm edges on bases, 8-cm height

f. triangular pyramid: 5-cm edges on bases, other edges 5 cm

g. square pyramid: 5-cm edges on bases, other edges 8 cm

h. pentagonal pyramid: 5-cm edges on bases, other edges 8 cm

i. hexagonal pyramid: 5-cm edges on bases, other edges 8 cm

6 You learned earlier in this lesson that at least three faces and at least three edges must meet at any vertex of a polyhedron. There is a deeper and more surprising relationship among the numbers of vertices, faces, and edges in any polyhedron.

a. Complete a table, like the one below, that includes some of the polyhedra for which your class constructed models.

	Number of Vertices	Number of Faces	Number of Edges
Cube			
Triangular Prism			
Hexagonal Prism			
Triangular Pyramid			
Square Pyramid			

b. Examine the table you made. Describe one or two patterns that you see.

c. Make a conjecture about the relationship among the numbers of faces F, vertices V, and edges E for each of these polyhedra. Compare your conjecture with others and resolve any differences.

d. Test the relationship you discovered in Part c using one of your other polyhedron models. Does it work for that polyhedron, too?

The formula relating the numbers of vertices, faces, and edges of convex polyhedra was first discovered by Swiss mathematician Leonhard Euler (1707–1783) and is called **Euler's Formula for Polyhedra**. The formula applies to all convex polyhedra.

7 Use Euler's Formula for Polyhedra to answer the following questions.

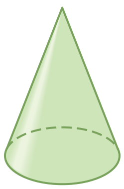
a. If a convex polyhedron has 10 vertices and 8 faces, how many edges does it have?

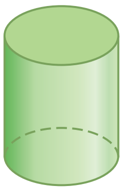
b. Can a convex polyhedron that has 8 faces, 12 edges, and 7 vertices be constructed? Give reasons for your answer.

8 *Cones* and *cylinders* are two other common three-dimensional shapes.

a. Are cones and cylinders polyhedra? Why or why not?

b. How do cones and cylinders compare to one another? How are they similar to and different from pyramids and prisms?


Cone


Cylinder

Summarize the Mathematics

In this investigation, you explored characteristics of commonly occurring three-dimensional shapes—prisms, pyramids, cylinders, and cones.

a In what ways are polyhedra like polygons? In what ways are they different?

b What is the least number of faces that can meet at a vertex of any polyhedron? What is the least number of edges that can meet at a vertex of any polyhedron?

c How are pyramids like prisms? How are they different?

d Consider a sequence of prisms in which each base is a regular polygon. The base of the first prism has 3 sides, the base of the second has 4 sides, the base of the third has 5 sides, and so on. As the number of sides in the base increases, what shape does the prism begin to resemble?

e Consider a sequence of pyramids with bases like those described in Part d. As the number of sides in the base of a pyramid increases, what shape does the pyramid begin to resemble?

f What formula relates the number of vertices, faces, and edges of the polyhedra that you explored in this investigation?

Be prepared to share your ideas and formula with the class.

Check Your Understanding

Examine the photo of a cereal box shown at the right.

a. Explain why the box is an example of a polyhedron. Then name the polyhedron as precisely as you can.

b. Draw a net for a model of the box.

c. Betsy claimed that this three-dimensional shape is a prism and that any of its faces could be considered to be its base. Do you agree with Betsy? Explain why or why not.

d. If you were to make a model of this three-dimensional shape from straws and pipe cleaners:

  i. how many different length straws would you need?

  ii. how many straws of each length would be needed?

e. Verify that Euler's Formula is satisfied for the polyhedron represented by the cereal box.

Geoff Butler

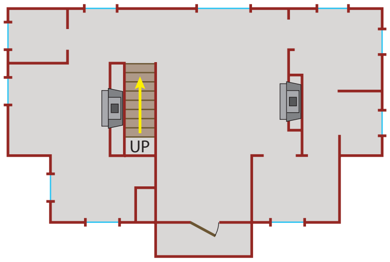
Investigation 2 Visualizing and Sketching Three-Dimensional Shapes

It is not always practical to construct models of three-dimensional shapes. For example, you cannot fax a scale model of an off-shore oil rig to an engineer in another country. Rather, the three-dimensional shape needs to be represented in two dimensions in a way that conveys the important information about the shape. In this investigation, you will explore these two main questions:

What are some effective ways to sketch three-dimensional shapes?

What information does each kind of sketch provide about the shape?

There are several methods for representing a three-dimensional shape in a sketch, but since the sketch has only two dimensions, some information about the three-dimensional shape will necessarily be missing. One way to depict three-dimensional shapes is to sketch two-dimensional *face-views* such as a top view, a front view, and a right-side view. Architects commonly use this method, called an **orthographic drawing**. For the house below, a *top view*, a *front view*, and a *right-side view* are shown. Together, these views display the length, depth, and height of the building to scale. You'll notice the top view is different from the other two. Floor plans such as this are frequently used instead of an exterior top view.

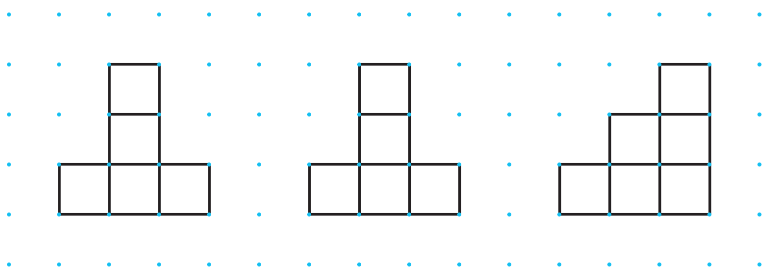


Top View

Front View

Right-Side View

1. An orthographic drawing of a model of a hotel made from cubes is shown below.



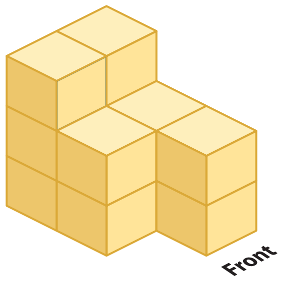
Top View Front View Right-Side View

a. How many cubes make up the model?

b. Use blocks or sugar cubes to make a model of this hotel. Build your model on a sheet of paper or poster board that can be rotated.

c. Could you make the model using information from only two of these views? Explain.

2 Examine this model of a building built from cubes. Assume any cube above the bottom layer rests on another cube and that there are no other hidden cubes.

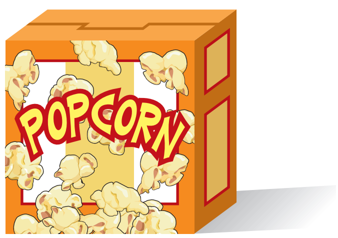


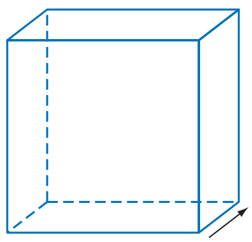
a. Make an orthographic sketch of this model.

b. How many cubes are in this model?

c. Would it be possible to make a model with fewer cubes that has the same top, front, and right-side views as this one? Explain.

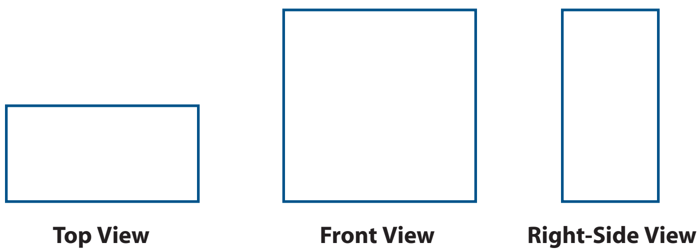
3 Another way to represent a three-dimensional shape such as a popcorn box is shown below. The sketch on the right, called an **oblique drawing**, is a *top-front-right corner* view of the box as a geometer would draw it. The front face was translated in the direction of the arrow to produce the back face, then edges were drawn to connect vertices. The sketch gives a sense of depth even though it is not drawn in true perspective. The three edges of the box blocked from view are shown as dashed lines.





a. What three-dimensional shape is the popcorn box?

b. The actual box is 10 inches high. Three face-views of the box drawn to scale are given below. Find the actual length and width by making appropriate measurements.



c. Now examine more carefully the sketch of the box from a top-front-right corner view.

i. What appear to be the shapes of the faces as shown in the drawing? What are the shapes of the faces in the real box?

ii. What edges are parallel in the real box? Are the corresponding edges in the sketch drawn parallel?

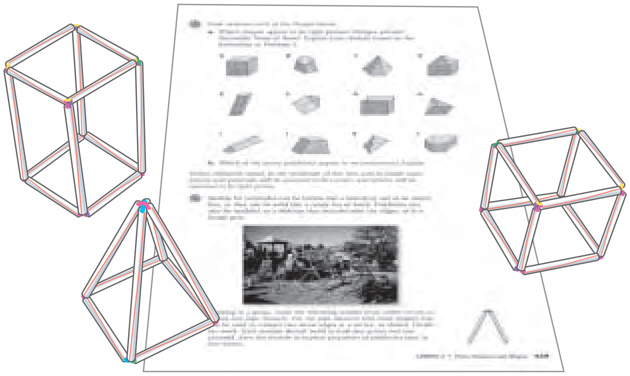
d. Sketch the box from a bottom-front-left corner view. Use dashed lines to show "hidden" edges.

4 For this problem, refer to the straw and pipe cleaner models you previously made. For each of the following models, place the model on your desk so that an edge of a base is parallel to an edge of your desk. Make an oblique sketch of the model. Compare your sketches and strategies for drawing these with those of your classmates.

a. cube

b. square prism

c. square pyramid



Summarize the Mathematics

A three-dimensional shape can be represented in two dimensions in various ways, including an orthographic (face-views) drawing or an oblique sketch from a particular point of view.

a When is it helpful to represent a three-dimensional shape by an orthographic drawing? By an oblique sketch?

b Discuss the similarities and differences between a top-front-right corner sketch of a right rectangular prism and the rectangular prism itself.

c Consider a convex polyhedron that is made up of two square pyramids sharing a common base. Make an orthographic drawing of this polyhedron. Assume an edge of the common base is parallel to an edge of your desk.

Be prepared to share your ideas and drawing with the class.

Check Your Understanding

Consider the three-dimensional shape formed when a pentagonal pyramid is placed on top of a pentagonal prism. Assume the bases of the two shapes are congruent.

a. Make an orthographic drawing of this shape.

b. Is the shape a convex polyhedron? Explain.

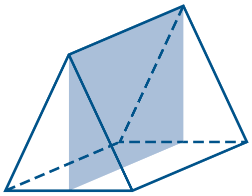
c. Describe a possible real-world application of a shape with this design.

Investigation 3 Patterns in Polyhedra

Like polygons, polyhedra are most useful and interesting when they have certain regularities or symmetries. For example, an "A-frame" is a style of architecture sometimes used in building houses. This attractive shape has a balance, or symmetry, about a vertical plane that contains the top roof line. The symmetry plane splits the basic shape of the house into two parts that are mirror images of one another. In this investigation, you will explore symmetry and other properties of polyhedra. As you work on the following problems, look for clues to this general question:

How are properties of polyhedra such as symmetry and rigidity related to corresponding properties of polygons?

1 The basic shape of an A-frame house is a prism with isosceles triangle bases. In the diagram below, note how the vertical plane cuts the three-dimensional shape into two parts that are mirror images of each other. This plane is called a **symmetry plane** or **mirror plane**. The shape is said to have **reflection symmetry**.

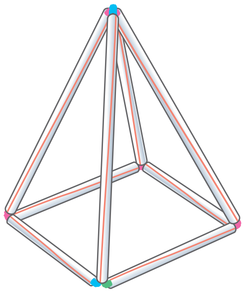


a. Does this isosceles triangular prism have any other symmetry planes? If so, describe or sketch them. If not, explain why not.

b. Next examine the cube and equilateral triangular pyramid models that your class made with straws and pipe cleaners. How many symmetry planes does each shape have? Describe or sketch them.

c. How are the symmetry planes for these three polyhedra related to the symmetry of their faces?

2 Place a model of a square pyramid on the top of a desk or table. Rotate it about the line through its apex and the center of its base.



a. What is the smallest angle that you can rotate the pyramid so that it appears to be in the same position as it was originally?

b. As you rotate the pyramid through a complete 360° turn, at what other *angles of rotation* does it appear as it did in its original location?

c. Can you rotate the pyramid through angles less than 360° about other lines so that it appears to be in the same position as originally? If so, describe them. If not, explain why not.

If there is a line about which a three-dimensional shape can be turned less than 360° in such a way that the rotated shape appears in exactly the same position as the original shape, the shape is said to have **rotational symmetry**. The line about which the three-dimensional shape is rotated is called an **axis of symmetry** or **rotation axis**.

3 Next examine the rotational symmetry of the equilateral triangular pyramid you constructed.

a. How many axes of symmetry are there? Where are the axes of symmetry located?

b. What are the angles of rotation for each axis of symmetry?

c. How is the rotational symmetry of the triangular pyramid similar to that of the square pyramid, and how is it different?

4 Now examine the rotational symmetry of the cube model you constructed.

a. How many axes of symmetry does the cube have? Where are the axes of symmetry located?

b. What are the angles of rotation for each axis of symmetry?

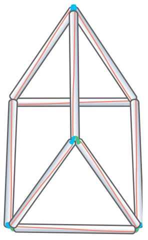
c. How is the symmetry of the faces related to the symmetry of the cube?

In addition to symmetry, another important consideration in the design of structures is *rigidity*. In Lesson 1, you discovered that any triangle is rigid; and that polygons with more than three sides can be made rigid by triangulating or bracing them with diagonals. In the next problem, you will investigate the rigidity of three-dimensional shapes.

5 Consider the models that you made in Problem 5 on page 429.

a. Which of the models represent rigid shapes?

b. Add bracing straws to your model of a triangular prism to make it rigid.

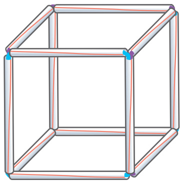


i. Describe where you placed the bracing straws and why you placed them there.

ii. Could you have placed the braces in different positions and still made the triangular prism rigid? Could you have used fewer bracing straws? Explain and illustrate.

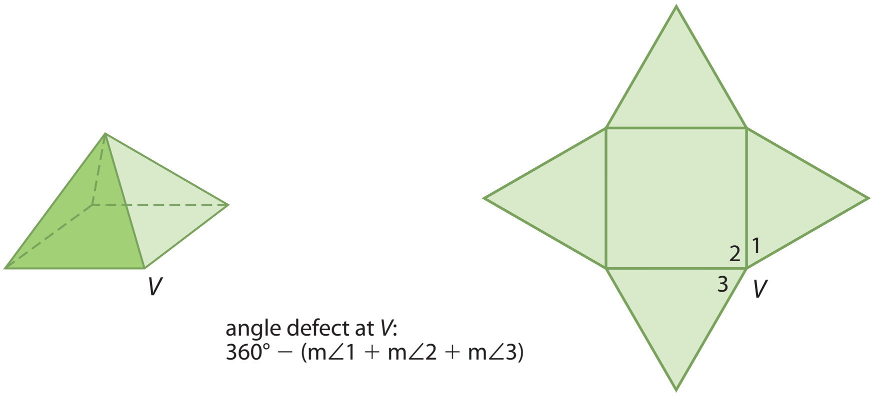
c. Add braces to your model of a cube so that it becomes a rigid structure. Note the number of bracing straws that you used and describe the position of each straw. Could you have used fewer bracing straws?

d. Think of a different way to reinforce the cube so that it becomes a rigid structure. Describe the pattern of reinforcing straws.



e. Of the methods you used to reinforce the cube, which could best be used to make a rectangular prism-shaped building stand rigidly?

6 Another interesting property of convex polyhedra has to do with the face angles that meet at each vertex. You saw in Investigation 1 that the sum of the measures of the face angles that meet at any vertex of any convex polyhedron must be less than 360°. The positive difference between the angle sum at a vertex and 360° is called the **vertex angle defect**. It is a measure of how close that corner is to being flat.



a. What is the angle defect at each vertex of a cube? What is the sum of the angle defects at all vertices of a cube?

b. In a similar way, find the sum of the angle defects for an equiangular triangular pyramid.

c. Find the sum of the angle defects for an equiangular triangular prism and for a regular hexagonal prism. Share the work with a partner.

d. Make a conjecture about the sum of the angle defects in any convex polyhedron. Test your conjecture using a square pyramid. Does your conjecture hold true in this case, too?

e. Compare your investigation of the face angles and angle defects in convex polyhedra to what you learned in Lesson 2 about angles of convex polygons. How are the ideas and the results alike? How are they different?

French philosopher and mathematician René Descartes (1596–1650) first discovered that the sum of the angle defects of any convex polyhedron is a constant. The result is called **Descartes' Theorem**.

Summarize the Mathematics

In this investigation, you explored two types of symmetry in three dimensions: reflection symmetry and rotational symmetry. You examined the rigidity of different polyhedra. You also discovered a property about the sum of the angle defects of a polyhedron.

a. Describe how to identify reflection symmetry and rotational symmetry in a polyhedron.

b. Name the rigid polyhedron with the fewest faces and edges.

c. What methods can be used to make a polyhedron rigid?

d. What is true of the sum of the vertex angle defects for the polyhedra that you studied in this investigation?

Be prepared to share your ideas with the entire class.

Check Your Understanding

Consider a polyhedron with all edges congruent that looks like a square pyramid joined to the top of a cube.

a. Is this a convex polyhedron? Explain.

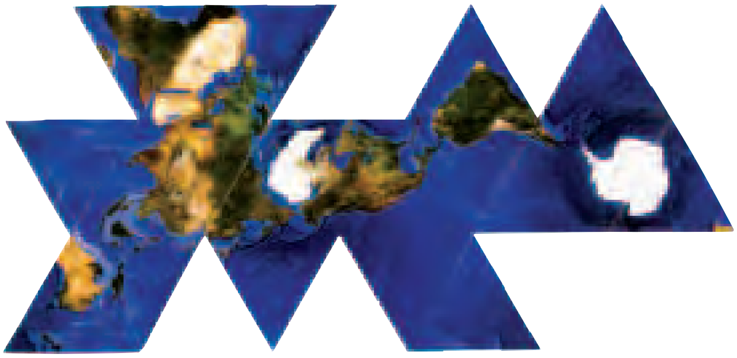
b. Describe the reflection symmetry and rotational symmetry of the shape.

c. Describe two ways to add bracing to make this polyhedron rigid. What is the least number of braces needed? Explain.

d. Calculate the sum of the angle defects for this polyhedron and verify that Descartes' Theorem is satisfied.

Investigation 4 Regular Polyhedra

In Lesson 2, you studied regular polygons and their properties. The three-dimensional counterpart of a regular polygon is a regular polyhedron. A **regular polyhedron**, also called a **Platonic solid**, is a convex polyhedron in which all faces are congruent, regular polygons. Furthermore, the arrangement of faces and edges is the same at each vertex. The regular polyhedron "globe" of the Earth shown here was created by R. Buckminster Fuller, inventor of the geodesic dome.



Recall that there are infinitely many different regular polygons, named by the number of sides—regular (or equilateral) triangle, regular quadrilateral (or square), regular pentagon, regular hexagon, and so on. You might think that there would also be infinitely many regular polyhedra. However, that is not the case! That fact is one of the more famous results in the history of geometry. In this investigation, you will explore these two related questions:

How many differently shaped regular polyhedra are possible and why?

What are some of the properties of these polyhedra?

1. Refer to the 9 polyhedra models you constructed in Investigation 1 using straws and pipe cleaners.
   - **a.** Which two models represent *regular* polyhedra? Explain why.
   - **b.** For each of these regular polyhedra, how many faces meet at each vertex?

2. To see whether other regular polyhedra can be constructed, begin by exploring how three or more congruent regular polygons can be arranged at a vertex of a polyhedron.
   - **a.** What is the sum of measures of the face angles that meet at any vertex of a regular triangular pyramid? Draw a partial net to illustrate your answer.
   - **b.** Next suppose 4 equilateral triangles meet at a vertex. If a regular polyhedron with vertices like this could be constructed, what would be the sum of the face angles at each vertex?
   - **c.** Repeat Part b for the case when 5 equilateral triangles meet at a vertex.
   - **d.** Explain why it is impossible for more than 5 equilateral triangles to meet at a vertex of a regular polyhedron.

3 Next, suppose the faces of a regular polyhedron have more than three edges. They might be squares, regular pentagons, regular hexagons, and so on.

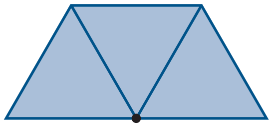
a. What is the sum of the measures of the face angles that meet at any vertex of a cube? Illustrate with a partial net.

b. Why is it not possible for 4 or more squares to meet at a vertex of a regular polyhedron?

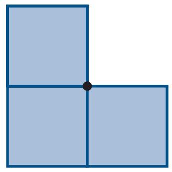
c. Discuss whether 3 regular pentagons, 3 regular hexagons, or 3 regular septagons could meet at a vertex of a regular polyhedron.

d. Explain why it is impossible for a regular polyhedron to have faces with more than 5 sides.

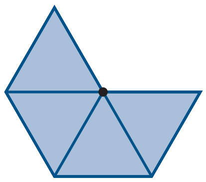
4 Putting together the steps of your reasoning in Problems 2 and 3, you have shown that there are at most 5 differently shaped regular polyhedra. The partial nets below show the different ways that regular polygon faces could meet at a single vertex of a regular polyhedron.



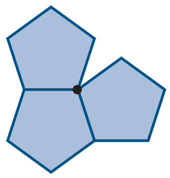
I



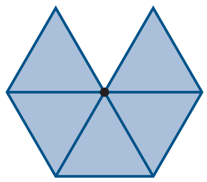
II



III



IV



V

a. Explain as precisely as you can why the above arrangements are the only ones possible.

b. Your reasoning likely has much in common with the reasoning in Lesson 2 that showed that the only regular polygons that tile the plane are the regular quadrilateral, regular triangle, and regular hexagon. How are the two arguments similar? How are they different?

c. For each of the partial nets above, assume that a regular polyhedron can be constructed for which the faces meet at each vertex as illustrated. What is the angle defect at each pictured vertex? How many vertices must each of the regular polyhedra have?

5 You have already constructed models of two of the five regular polyhedra, namely, an equilateral triangular pyramid, also called a **regular tetrahedron** (named from the Greek "tetra" for its 4 faces), and the cube, also called a **regular hexahedron** (named for its 6 faces).

a. Collaborate with your classmates to construct models for a single "corner" of a regular polyhedron using copies of the third, fourth, and fifth partial nets shown in Problem 4. Each student should cut out one partial net, fold, and close the corner by joining two edges with tape. Compare your model to those of classmates who used the same net. Are the models based on the same net identical? If not, resolve the differences.

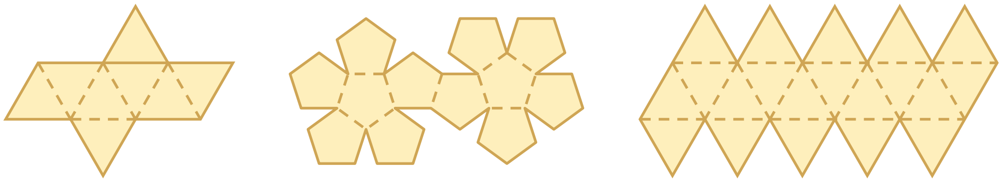
b. Working in pairs, select either partial net III or IV in Problem 4 to complete the following tasks.

i. Tape together as many partial nets as needed to form a model for a regular polyhedron. How many copies did you use?

ii. Describe the polyhedron you formed. How many faces does it have? What would be a good name for the polyhedron? Why?

c. Use the same procedure as in Part b to construct a model for a regular polyhedron using multiple copies of the fifth partial net in Problem 4. Discuss what happens.

d. Although it cannot be constructed only from copies of its "corners" where 5 faces meet, there is a fifth regular polyhedron called a **regular icosahedron** (from the Greek "eikosi" meaning 20). Nets for the last three regular polyhedra are shown below. Construct a model of a regular icosahedron by cutting out a copy of its net and folding and taping.



Octahedron **Dodecahedron** **Icosahedron**

6 As you can see from examination of the models, regular polyhedra have many symmetries. You previously explored symmetries of a regular tetrahedron and a regular hexahedron. Study your model of a regular octahedron.

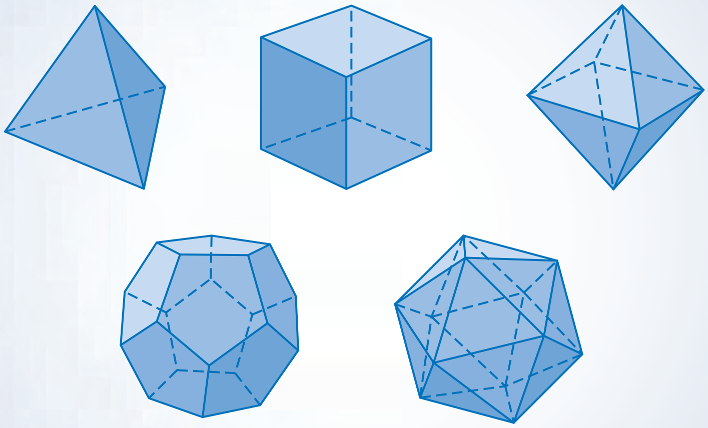
a. How many planes of symmetry does it have? Describe their locations.

b. As for rotational symmetry, a regular octahedron has 6 axes of 180° symmetry, 4 axes of 120° symmetry, and 3 axes of 90° symmetry. Describe the locations of the axes for each of these three types of rotational symmetry.

c. How do the types and number of axes of symmetry for the regular octahedron compare with those of the cube?

Summarize the Mathematics

In this investigation, you demonstrated that there are exactly 5 differently shaped regular polyhedra and examined some of their symmetries.

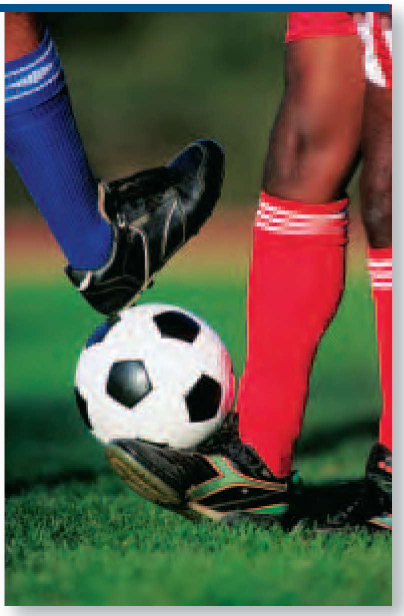


a Name all the regular polyhedra according to the number of faces of each. For each regular polyhedron, describe a face and give the number of faces that meet at each vertex.

b Explain why there cannot be more than 5 differently shaped regular polyhedra.

Be prepared to share your descriptions and explanation with the entire class.

Check Your Understanding



The design of a soccer ball is based on a **semi-regular polyhedron**; that is, a polyhedron with faces that are congruent copies of two or more different regular polygons. As in a regular polyhedron, the arrangement of faces and edges is the same at each vertex.

a. Describe the three faces that meet at each vertex of the polyhedron that is used for the soccer ball.

b. What is the sum of the three face angles that meet at each vertex of the polyhedron? What is the angle defect at each vertex?

c. Find the number of vertices of the polyhedron.

d. The polyhedron consists of 20 hexagons and 12 pentagons. How many edges does it have?

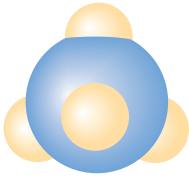
On Your Own

Applications

1. Three-dimensional shapes are the basis of atomic structures as well as of common structures for work, living, and play. Often three-dimensional shapes are assembled from a combination of simpler shapes.

   a. Study this photograph of Big Ben in London, England. What three-dimensional shapes appear to be used in this tower? Which are prisms? Which are pyramids?

   b. Scientists use three-dimensional structures to model molecules of compounds, such as the model of a methane molecule shown below. Describe and name the polyhedron with the skeleton that would be formed by joining the outermost points of the four hydrogen atoms that are equally spaced around the central carbon atom.



Big Ben clock tower in London, England

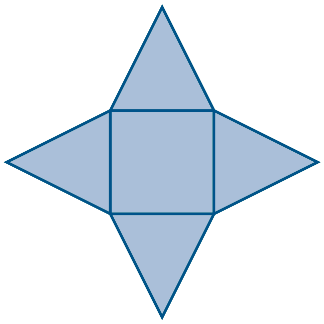
2. Make a conjecture about the relationship between the circumference of a circular column and the weight it can support.

   a. Conduct an experiment to test your conjecture about the weight-bearing capability of circular columns. Use columns of the same height but with different circumferences.

   b. Organize your data in a table and display them in a graph.

   c. What appears to be true about the relationship between the circumference of a column and the weight it can support? Why do you think this happens?

3. A net for a square pyramid is shown at the right. Lateral faces are equilateral triangles.



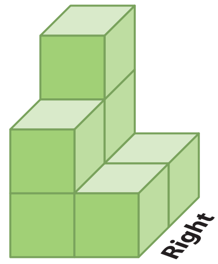
   a. Sketch two other nets that would fold into this pyramid.

   b. How many straight cuts are needed to cut out the net at the right? Each of your nets?

   c. Is there a net that requires even fewer straight cuts than the ones you have examined so far? Explain.

On Your Own

4 Building designers can test their designs by using identical cubes to represent rooms. They can use the cubes to try various arrangements of rooms. Study this drawing of a cube hotel.

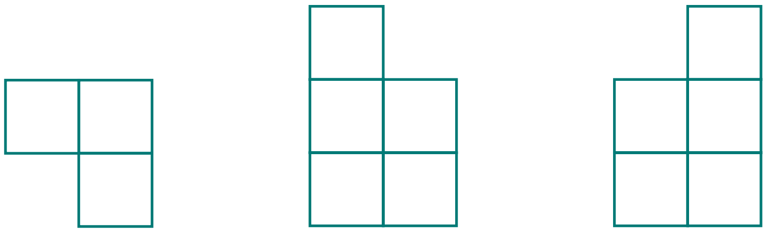


a. How many cubes are there in the model? Assume any cube above the bottom layer rests on another cube.

b. Draw the top, front, and right-side orthographic views of this shape.

c. Is the model pictured on the right a polyhedron? If not, explain why not. If so, is it convex or nonconvex?

5 Three views of a cube model of a hotel are shown below.



a. Make an oblique drawing of this hotel model from a vantage point that shows clearly all the characteristics of the model. Assume any cube above the bottom layer rests on another cube.

b. Is there more than one model with these three views? If so, make a drawing of a second one from a vantage point that illustrates how this model differs from the one in Part a.

c. How many cubes are there in each hotel model?

6 Both portability and rigidity are design features of a folding "director's chair."

a. How are these features designed into the chair shown at the right?

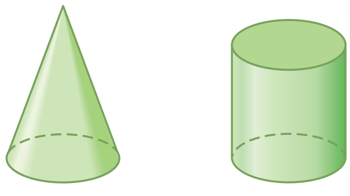
b. The pair of legs at the front and back are attached at their midpoints. Draw and label a diagram of the front pair of legs and edge of the seat. Using congruent triangles and Connections Task 18, page 391, explain as carefully as you can why these conditions guarantee that the outstretched seat will be parallel to the ground surface.

c. Identify two other commonly used items that can collapse but must remain rigid when "unfolded." Analyze their designs.

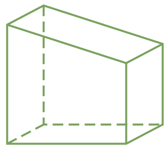
7 Symmetry can help to describe and also to construct a three-dimensional shape.

a. Describe the reflection symmetry and the rotational symmetry of these three-dimensional shapes.

- **i.** a right circular cylinder
- **ii.** a right circular cone
- **iii.** a prism with a parallelogram base
- **iv.** a regular pentagonal pyramid
- **v.** a sphere



b. Half of a polyhedron is shown at the right.



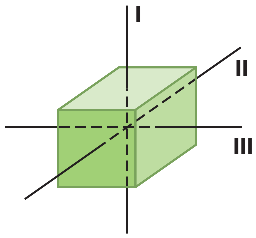
- **i.** Sketch the entire polyhedron if it is symmetric about the plane containing the right face.
- **ii.** Sketch the entire polyhedron if it is symmetric about the plane containing the left face.

c. Are the polyhedra you sketched in Part b convex? Explain your answers.

d. For each polyhedron in Part b, count the vertices, faces, and edges. Verify that Euler's Formula holds for each polyhedron.

e. Using a polyhedron from Part b above as an example, explain why Descartes' Theorem does not make sense for nonconvex polyhedra.

8 A model of a square prism could be made from a potato or modeling clay. Consider the $5 \times 5 \times 6$ square prism pictured below. The three lines (skewers) intersecting the prism contain the centers of opposite faces.



a. Explain why these lines are axes of symmetry for the prism.

b. What angles of rotation are associated with each axis of symmetry? Explain your reasoning.

c. Does this prism have other axes of symmetry? If so, describe their locations and give the angle of rotation associated with each.

d. How is the rotational symmetry of a cube similar to that of the square prism above? How is it different?

9 The square pyramids at Giza in Egypt are pictured here. The lateral faces are isosceles, but not equilateral, triangles.

a. Describe the planes of symmetry and the rotational symmetry of one of the Giza pyramids.

b. In the steepest of these pyramids, the face angles at the apex are each about 40°. Find the angle defect at each of the 5 vertices of this pyramid. Verify that Descartes' Theorem holds for this pyramid.

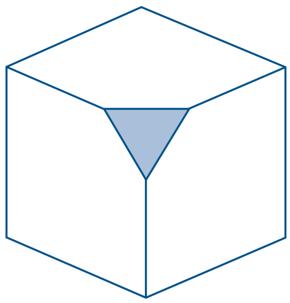
Connections

10 A pyramid has a square base that is 10 units on a side, and the other faces of the pyramid are congruent isosceles triangles.

a. Suppose you were going to make a model with straws and pipe cleaners of such a pyramid. Could the lateral edges to the apex be 5 units long? Could they be 10 units long? Could they be 20 units long?

b. Is there a minimum length for the lateral edges? Is there a maximum length for the lateral edges? If so, what are they? What property or properties from earlier lessons in this unit would justify your answers?

11 Imagine a model of a cube made of clay or cut from a potato. Make such a model if possible.



a. How many faces, edges, and vertices does a cube have?

b. Slice a corner off (as shown), making a small triangular face. How many faces, edges, and vertices does the new polyhedron have?

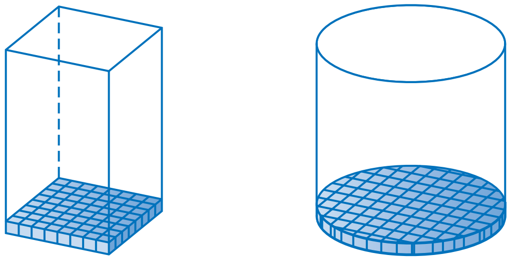
c. Repeat at each corner so that the slices do not overlap. Make a table showing the number of faces, edges, and vertices of the modified cube after each "corner slice."

d. Using *NOW* and *NEXT*, write a rule describing the pattern of change in the number of faces after a slice. Write similar *NOW-NEXT* rules for the number of edges and for the number of vertices after each slice.

e. Do you think Euler's Formula will hold at each stage? Justify your answer using the *NOW-NEXT* rules in Part d.

f. How many faces, edges, and vertices does the new polyhedron have when all the corners are sliced off?

12 When filling three-dimensional containers like boxes or cylindrical cans, a measure of *volume* is needed. The volume of a three-dimensional shape of height 1 unit is numerically equal to the area of the base. For prisms and cylinders, imagine the first layer of 1-unit cubes as shown below.



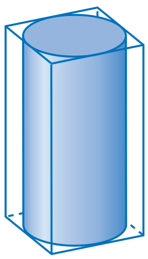
Then add additional layers until the prism or cylinder is filled. The number of layers is the shape's height. This suggests that the volume V of either a prism or a circular cylinder is the product of the *area of the base* B and the *height* h, a formula usually written symbolically as $V = Bh$.

a. Find the volume of a square prism with base edges of 7 cm and height of 10 cm. The volume is in what units?

b. Find the volume of a cylinder in which the radius of the base is 5 in. and the height is 9 in. Indicate the units.

c. Find the volume of a regular triangular prism if all edges are 4 ft long.

d. A right circular cylindrical can is packed snugly into a box as shown here. The base of the box is a square 8 cm on a side and the height of the box is 16 cm.



i. Find the volume of the space between the can and the box.

ii. Find the ratio of the volume of the cylinder to the volume of the box. What does the ratio tell you?

e. Suppose the length of the side of a square box as in Part d is s cm and the height is h cm.

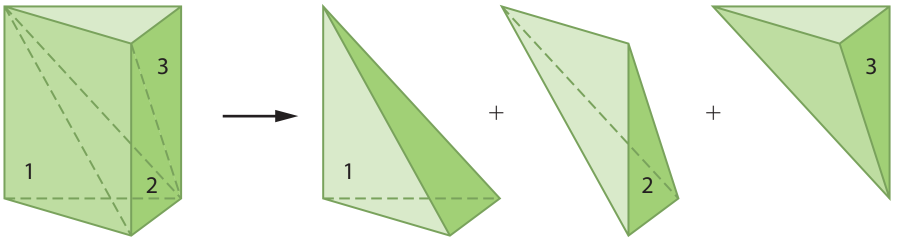
i. What is the volume of the box in terms of s and h?

ii. What are the radius and height of the cylinder in terms of s and h?

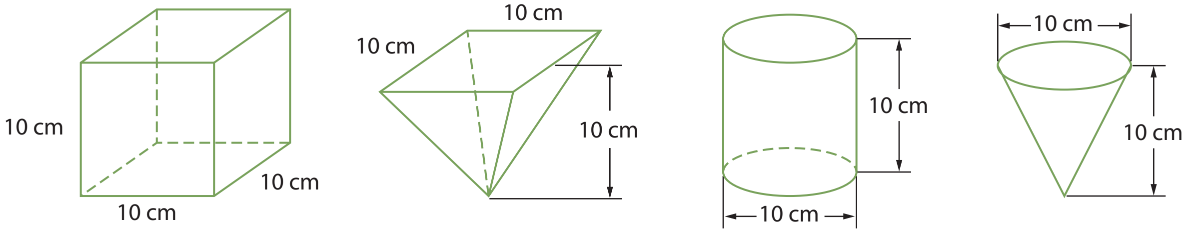
iii. What is the volume of the cylinder in terms of s and h?

iv. What is the ratio of the volume of the cylinder to the volume of the box? What does the ratio tell you?

13 As seen in Connections Task 12, an important volume formula that holds for both prisms and cylinders is $V = Bh$ where B is the area of the base and h is the height, that is, the perpendicular distance between the bases. A second related formula, $V = \frac{1}{3}Bh$, holds for pyramids and cones. The illustration below shows a triangular prism dissected into three triangular pyramids of equal volume. This explains the use of the fraction $\frac{1}{3}$ in the formula $V = \frac{1}{3}Bh$.



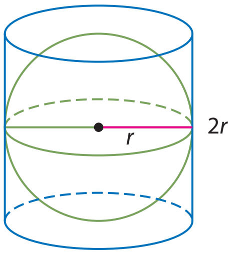
a. Find the volume of the cube, pyramid, cylinder, and cone shown below. How do the volumes of these shapes compare?



b. A movie theater sells different sizes of popcorn in different shaped containers. One is a cylinder with a height of 17 cm and radius of the base of 8 cm. It sells for \$4.50. Another is a rectangular box (prism) with width 10 cm, length 16 cm, and height 18 cm. It sells for \$3.75. Which is the best buy? Explain your answer.

c. A third popcorn container is a cone with a height of 24 cm and radius of the base of 10 cm. What is the most its price could be if it is the best buy of the three containers?

14 In three dimensions, the counterpart of a circle is a *sphere*. To discover a formula for the volume of a sphere, consider the following experiment. Imagine or obtain a sphere of radius r that fits snugly inside a cylinder of diameter $2r$ and height $2r$ as shown. Remove the hollow sphere and fill it with water. Pour the water into the cylinder.



a. Write an expression for the volume of the cylinder in terms of π and r.

b. A class at the Battle Creek Mathematics and Science Center conducted the experiment with different-sized spheres and cylinders. In each case, the height of the water in the cylinder was about $\frac{4}{3}r$. Use the results of their experiment to write a formula for the volume V of a sphere in terms of its radius r.

15 Use your regular polyhedron models and the nets to complete a table like the one below.

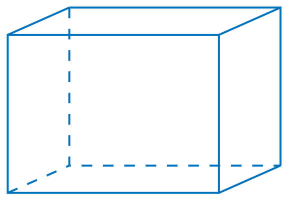
	Edges per Face	Faces per Vertex	Number of Faces	Number of Vertices	Number of Edges
Tetrahedron		3	4		
Hexahedron		3		8	
Octahedron		4			12
Dodecahedron		3	12		
Icosahedron		5	20		

a. Describe at least two interesting patterns that you see in the table.

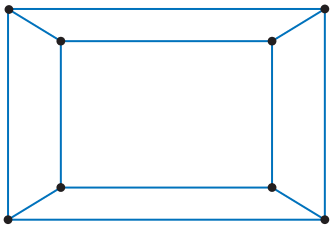
b. Check Euler's Formula for these polyhedra.

c. Eric conjectured that the number of edges in a regular polyhedron is the product of the number of edges per face and the number of faces. Do you agree with Eric? If so, explain why. If not, explain how to correct Eric's statement.

16 In this lesson, you learned how to represent three-dimensional shapes in two dimensions with orthographic (face-view) drawings and with oblique sketches. For the case of a rectangular prism, another way to represent the shape is illustrated below.



Rectangular Prism

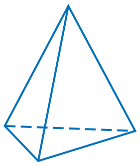
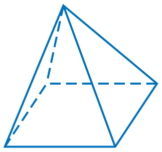
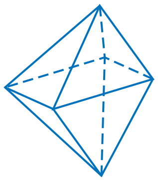


Vertex-Edge Graph

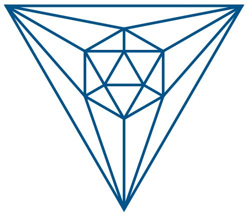
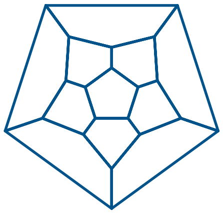
a. The figure on the right is a representation of a rectangular prism as a vertex-edge graph with no edge crossings. (When there are no edge crossings, the graph is called a *planar graph*.) You can think of this graph, called a **Schlegel diagram**, as resulting from "compressing" a rectangular prism with elastic edges down into two dimensions. (You can also think of the graph as resulting from a one-point perspective projection, where the vanishing point is in the center of the back face. Think of looking at a clear box with your nose very close to the front face.) The graph has lost most of the shape of the three-dimensional prism, but it shares many of the prism's properties. Name as many shared properties as you can.

b. How many faces does the rectangular prism have? Explain how all the faces are accounted for in the regions formed by the vertex-edge graph.

c. Use the idea of *compressing* illustrated in Part a to draw Schlegel diagrams for the following polyhedra. After compressing (or projecting) the octahedron, nonintersecting edges in the octahedron should not intersect in the graph. Imagine projecting the model while looking directly at the center of one face.

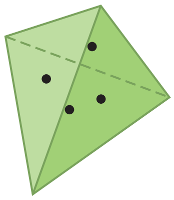
i. 
ii. 
iii. 

d. Here are Schlegel diagrams for the regular dodecahedron and the regular icosahedron. Which is which? Explain your reasoning.

i. 
ii. 

17 There are interesting and useful connections between pairs of regular polyhedra.

a. Refer to the regular tetrahedron pictured on the right. The center of each face is marked.



- **i.** How many such centers are there?
- **ii.** Imagine connecting the centers with segments. How many such segments are there?
- **iii.** Visualize the polyhedron having these segments as its edges. What are the shapes of its faces?
- **iv.** What polyhedron is formed by connecting the centers of the faces of a tetrahedron?

b. Now using a model of a cube, imagine the center of its faces.

- **i.** How many such centers are there?
- **ii.** Imagine connecting with segments, each center to the center of the four *adjacent faces*. Adjacent faces are faces that have a common edge. How many such segments are there?
- **iii.** Visualize the polyhedron having these segments as its edges. What are the shapes of its faces? How many faces are there?
- **iv.** What polyhedron is formed by connecting the centers of adjacent faces of a cube?

c. What happens if you start with the new polyhedron in Part b and repeat the process of connecting centers of adjacent faces with segments?

d. Make a conjecture about the polyhedron formed by connecting the center of adjacent faces of a regular dodecahedron. Explain the basis for your conjecture.

Reflections

18 Compare the definitions of convex polygon (page 404) and convex polyhedron (page 426). If all the faces of a polyhedron are convex polygons, must the polyhedron be convex? Explain.

19 Isaiah made the following two conjectures about the symmetry of a right prism. Indicate whether you agree or disagree with each, and write arguments in support of your positions. *Hint:* If you think a statement is false, one counterexample (that is, an example of a right prism that does not have the named symmetry) is a sufficient argument. A more general argument based on the properties of a right prism is required if you think a statement is true.

a. Every right prism has at least one symmetry plane.

b. Every right prism has at least one axis of rotational symmetry.

20 In this unit, as in previous units, you have engaged in important kinds of mathematical thinking. Look back over the four investigations in this lesson and consider some of the mathematical thinking you have done. Describe an example where you did each of the following:

a. experiment

b. search for and explain patterns

c. formulate or find a mathematical model

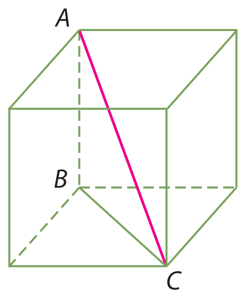
d. visualize

e. make and check conjectures

f. make connections between geometry and algebra

Extensions

21 The faces of a cube are congruent squares. You know how to use the length of each edge of such a cube to find the lengths of the diagonals on the faces. Cubes also have "body" diagonals such as $\overline{AC}$ in the diagram at the right.



a. Find the length of the body diagonal $\overline{AC}$ of a cube when the cube edges are as given below. Express your answers in simplest radical form.

i. 1 inch long **ii.** 2 inches long **iii.** 3 inches long

b. How many body diagonals does a cube have? Explain as precisely as you can why all body diagonals are the same length.

c. Find a formula for calculating the length of each body diagonal in a cube with edge length x.

22 Analysis of the formulas for the volume of a prism and for the volume of a cylinder suggests that multiplying the dimensions of the shape by a positive constant changes the volume in a predictable way.

a. One large juice can has dimensions twice those of a smaller can. How do the volumes of the two cans compare?

b. One cereal box has dimensions 3 times those of another. How do the volumes of the two boxes compare?

c. If the dimensions of one prism are 5 times those of another, how do the volumes compare?

d. If the dimensions of one prism are k times those of another, how do the volumes compare?

23 In Lesson 2, you identified polygons that tile the plane. In three-dimensional geometry, the related question is: What three-dimensional shapes will fill space? An obvious example of a space-filling, three-dimensional shape is a rectangular prism. In fact, the efficiency with which rectangular prisms can be stacked is what makes their shape so useful as boxes and other containers.

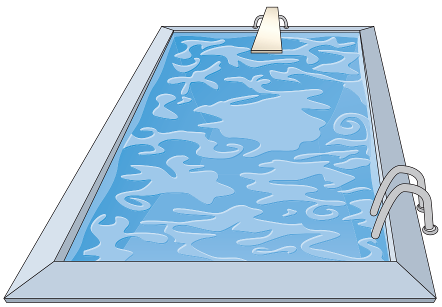
a. What right prisms with regular polygonal bases will fill space? Explain how your answer is related to the regular polygons that will tile the plane.

b. The cells of a honeycomb are approximated by regular hexagonal prisms, which form a three-dimensional tiling for storing honey. Suppose the perimeter of the base of one cell of a honeycomb is 24 mm and the height is 20 mm. What is the *lateral surface area* (surface area not including top and bottom) of a single cell? Explain.

c. Which of the three right prisms with regular polygonal bases that fill space (see Part a) produces the cell with the greatest volume when the perimeter of the base is 24 mm and the height is 20 mm? As the number of sides of the base of a prism with a regular polygonal base of fixed perimeter increases, how does the corresponding volume change?

d. Write a statement summarizing how three-dimensional shape is an important factor in the building of the cells of a honeycomb.

24 A rectangular swimming pool is 28 feet long and 18 feet wide. The shallow end is 3 feet deep and extends for 6 feet. Then for 16 feet horizontally, there is a constant decline toward the 9-foot-deep end.



a. Sketch the pool and indicate the measures on the sketch.

b. How much water is needed to fill the pool within 6 inches of the top?

c. One gallon of paint covers approximately 75 square feet of surface. How many gallons of paint are needed to paint the inside of the pool? If the pool paint comes in 5-gallon cans, how many cans should be purchased?

d. How much material is needed to make a rectangular pool cover that extends 2 feet beyond the pool on all sides?

e. About how many 6-inch square ceramic tiles are needed to tile the top 18 inches of the inside faces of the pool?

25 For each figure below, describe how a plane and a cube could intersect so that the intersection (or cross-section) is the figure described. If the figure is not possible, explain your reasoning.

a. a point

b. a segment

c. a triangle

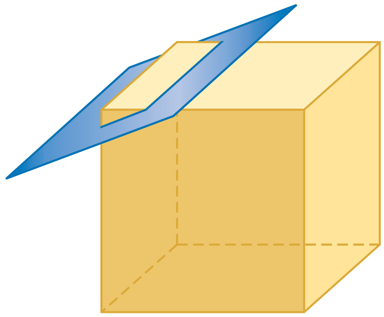
d. an equilateral triangle

e. a square

f. a rectangle

g. a pentagon

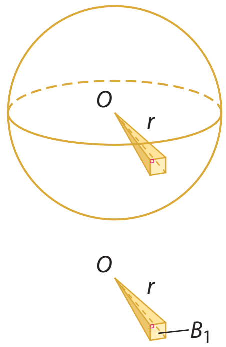
h. a hexagon



26 The volume formula for a sphere (see Connections Task 14) can be used to derive a formula for the *surface area* of a sphere. Consider a solid sphere with center at point O and radius r. Imagine a tiny polygon-like portion of the sphere's surface. This "polygon" is the base of a pyramid with vertex O and height approximately r.

a. If the area of the base of the polygon is B_1, what is the volume of the pyramid?

b. Next, imagine dividing the entire surface of the sphere into a large number n of tiny, nonoverlapping "polygons," with respective areas of $B_1, B_2, B_3, \ldots, B_n$. Write a formula for the surface area S in terms of $B_1, B_2, B_3, \ldots, B_n$.



c. Explain why a formula for the volume V of the sphere is

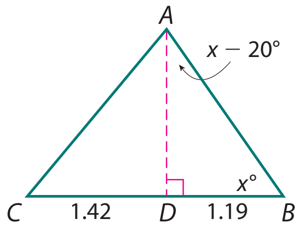
$$V = \frac{1}{3}(B_1)(r) + \frac{1}{3}(B_2)(r) + \cdots + \frac{1}{3}(B_n)(r).$$

d. Rewrite the formula in Part c by (i) substituting $\frac{4}{3}\pi r^3$ for V and (ii) factoring the largest common factor from the right-hand side.

e. Use your result from Part d to write a formula for the surface area S of a sphere in terms of its radius r.

Review

27 Examine the information given in the diagram below. Write and solve an equation to find the measure of $\angle ABD$.



28 Determine whether each of the following tables indicates a linear relationship between x and y. For those that are linear, find a rule that describes the relationship.

a.

x	0	1	2	5
y	1	5	9	21

b.

x	0	1	2	3
y	1	5	7	37

c.

x	−1	0	1	2
y	1	0	−1	−8

29 Use algebraic reasoning to solve each equation or inequality.

a. $150 = 20 - 6x$

b. $150 > 20 - 6x$

c. $6(x + 8) = -72$

d. $6(x + 8) \geq -72$

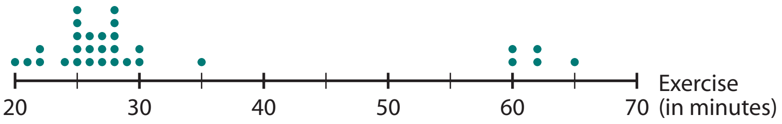
30 Complete each sentence with the name of a polygon.

a. Every __________ has exactly two lines of symmetry.

b. Every rhombus is also a _________.

c. The diagonals of every ________ are lines of symmetry.

31 Andy kept track of the number of minutes he exercised each day for the last 30 days. His data are shown on the dot plot below.



a. What was the greatest amount of time that Andy exercised in one day?

b. On how many days did Andy exercise for 25 minutes?

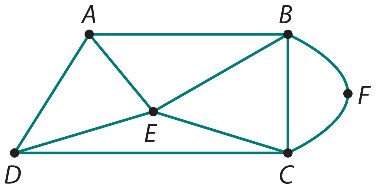
c. What percentage of the days did Andy exercise for at least one hour?

d. Find the median number of minutes that Andy exercised for these 30 days.

e. Would you expect the mean number of minutes exercised to be greater than or less than the median number of minutes exercised? Explain your reasoning.

32 Produce a graph of the function $y = x^{10}$, for $0 < x < 2$. On the same set of axes, produce a graph of the function $y = 10^x$. About where do the two graphs intersect?

33 Does the vertex-edge graph shown below have an Euler circuit or path? If so, find one. If not, describe how you know. (Remember that an Euler path traces all edges of the graph exactly once but is not a circuit.)



34 In almost all countries, temperature is measured in degrees Celsius. Temperature C in degrees Celsius is related to temperature F in degrees Fahrenheit by the formula

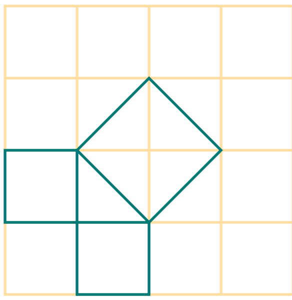
$$C = \frac{5}{9}(F - 32).$$

a. Describe, in words, how to convert from Fahrenheit to Celsius.

b. Write an equivalent formula that relates temperature C in degrees Celsius to temperature F in degrees Fahrenheit.

c. The temperature in New York City's Central Park on July 6, 1999, reached a high of 92° Fahrenheit. Express this temperature in degrees Celsius.

35 Make a copy of the diagram below on a piece of grid paper or square dot paper. The legs of the right triangle each have a length of 1 unit.



a. What are the areas of the squares shown on the sides of the triangle?

b. On your paper, draw a similar diagram to show two squares, each with area 4 square units and a square with area 8 square units.

c. Use a similar diagram to draw a square with area 5 square units.

d. On your diagrams indicate line segments with lengths $\sqrt{2}$, $\sqrt{8}$, and $\sqrt{5}$.

e. From your diagrams, estimate the lengths $\sqrt{2}$, $\sqrt{8}$, and $\sqrt{5}$.

LESSON 4

Looking Back

In this unit, you studied two- and three-dimensional shapes and how they are related. You learned how segment lengths and angle measures determine the shape of special polygons and how polygons determine the shape of polyhedra. You discovered ways to test triangles for congruence and how to use triangles to study useful properties of parallelograms and other polygons. You explored how to visualize three-dimensional shapes and how to represent them in two dimensions. You also learned about symmetry and other properties that make certain shapes useful in design, engineering, and construction. Finally, you learned how knowledge of a few basic properties of shapes could be used to reason to additional properties of those shapes.

The following tasks will help you review, pull together, and apply what you have learned in the process of solving several new problems.

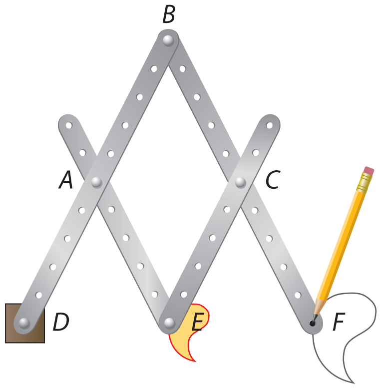
1 Suppose you are given an envelope containing information on separate slips of paper about each of the three side lengths (AB, BC, AC) and each of the three angle measures ($m\angle A$, $m\angle B$, $m\angle C$) of the triangular truss being carried by Diane and Jessie. You randomly draw one slip at a time from the envelope.

a. What is the *largest* number of slips you would ever need to draw before you had enough information to build a congruent truss? Explain your reasoning.

b. What is the *smallest* number of slips you could draw and still build the truss? Explain.

2 Some quadrilateral linkages can change rotary motion into "back-and-forth" motion and vice versa. In addition to being used in mechanical devices, the parallelogram linkage serves as the basis for a linkage called a *pantograph*. Pantographs are used for copying drawings and maps to a different scale.

The pantograph shown has been assembled so that $ABCE$ is a rhombus; $AD = CF$ and these lengths are the same as that of a side of the rhombus. The pantograph is held firm at point D. No matter how the linkage is moved, give reasons why:



a. $ABCE$ will always be a rhombus.

b. $DE = EF$

c. Points D, E, and F will always be on a straight line.

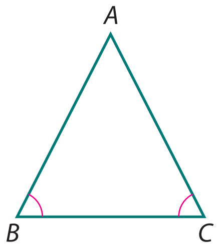
3 Earlier in the unit you saw that the base angles of any isosceles triangle are congruent. This property can be restated as:

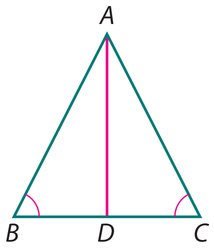
If two sides of a triangle are congruent, then the angles opposite those sides are congruent.

a. Write the converse of this statement. Do you think the converse is a true statement?

b. Describe an experiment you could conduct to test whether the converse *might* be true.

c. The first diagram below shows $\triangle ABC$ with two congruent angles. The second diagram shows the same triangle with the bisector of $\angle A$ drawn.

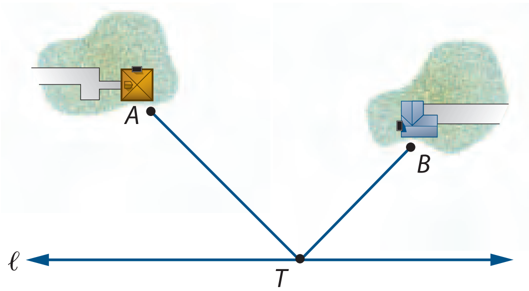




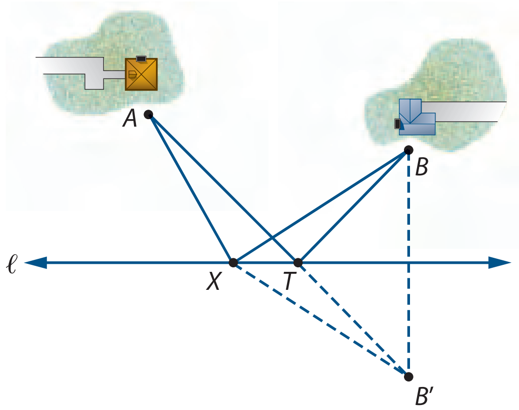
How could you use careful reasoning and the given information for the second diagram to show that the converse statement is true?

d. What would the experiment in Part b tell you about this situation? How does that differ from what the reasoning in Part c tells you?

4 Two farms, located at points A and B, are to be connected by separate wires to a transformer on a main power line ℓ.



Study the diagram below which shows a method for locating the position of the transformer.



Point B' is located so that the power line ℓ is the perpendicular bisector of $\overline{BB'}$. Then the location T of the transformer is determined by sighting the line AB'. By locating the transformer at point T rather than at any other point X on the line, the power company uses the minimum amount of wire to bring electricity to the two farms.

a. Why is the length of the required wire, $AT + TB$, the same as the length AB'?

b. Explain as carefully and precisely as you can why if any other location X is chosen, then more wire would be required.

5 Make an accurate drawing of a regular decagon.

a. What must be the measure of an interior angle?

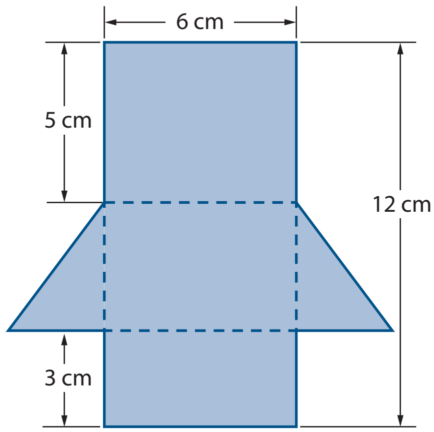
b. What must be the measure of an exterior angle?

c. Describe its lines of symmetry and its rotational symmetries.

d. Will a regular decagon tessellate a plane? Explain why or why not.

e. Will copies of an equilateral triangle, a regular decagon, and a regular 15-gon form a semiregular tessellation? Explain your reasoning.

6 Midland Packaging manufactures boxes for many different companies. The net for one type of box manufactured for a candy company is shown below.



a. Name the three-dimensional shape for which this is a net.

b. Sketch the box showing its hidden edges.

c. Sketch two other possible nets that could be used to manufacture the same box.

d. Find the volume of the box.

e. Find the surface area of the box.

f. Does the box have any symmetries? If so, explain how the symmetries are related to the symmetries of its faces.

7 A common basic shape for a house is a polyhedron formed by placing a triangular prism on top of a rectangular prism. For a particular house, the length, width, and height of the rectangular prism are 70 feet, 50 feet, and 20 feet, respectively. The base of the triangular prism is an isosceles triangle, and the angle at the peak of the roof is 140°.

a. The resulting polyhedron can be viewed as a single prism. Make a scale drawing of the base of the prism formed in this way.

b. Make an orthographic drawing of this prism.

c. Describe the reflection symmetry and the rotational symmetry of the prism.

d. Determine the number of vertices, edges, and faces for this prism. Verify that Euler's Formula holds in this case.

e. Find the angle defect at each vertex. Verify that Descartes' Theorem holds in this case.

8 Two-dimensional concepts often have corresponding, though not usually identical, concepts in three dimensions. Answer the following questions about some of these connections.

a. What property of polygons is related to Descartes' Theorem in three dimensions?

b. What is the three-dimensional counterpart of line symmetry in two dimensions?

c. How are rotational symmetry in two dimensions and rotational symmetry in three dimensions alike? How are they different?

d. What is the three-dimensional counterpart of tiling the plane in two dimensions?

e. What are the three-dimensional counterparts of a triangle, square, circle, rectangle, parallelogram, and regular polygon?

f. How is rigidity of two-dimensional shapes like rigidity of three-dimensional shapes? How is it different?

g. Identify at least one more two-dimensional concept or shape and describe its counterpart in three dimensions.

9 Consider the following two statements, one about shape in two dimensions, the other about shape in three dimensions.

- A regular hexagon tiles the plane.
- It is not possible for a regular polyhedron to have only regular hexagonal faces.

These two statements are true for very similar reasons. Explain why each statement is true, and explain their connections.

Summarize the Mathematics

Shape is a fundamental feature of the world in which you live. Understanding shape involves being able to identify and describe shapes, visualize and represent shapes with drawings, and analyze and apply properties of shapes.

a Triangles and quadrilaterals are special classes of shapes called polygons.

i. What properties are true of every polygon?

ii. What properties are true of every quadrilateral? What property of some quadrilaterals makes the shape widely useful as a linkage?

iii. What properties are true of every triangle?

b What does it mean for two polygons to be congruent?

i. What information is sufficient to test whether two triangles are congruent?

ii. Which test in part i could be used to test whether two parallelograms are congruent?

iii. How can you use the idea of triangle congruence to reason about properties of polygons and parallelograms in particular? What are some of these properties?

c If a statement is true, its converse may or may not be true. What is the converse of the Pythagorean Theorem? Explain why it is true. How is it used in applications?

d Polyhedra are three-dimensional counterparts of polygons.

i. Compare and contrast polygons and polyhedra.

ii. Describe a variety of ways that you can represent polyhedra.

iii. In some cases, congruent copies of a polygon can be used to tile a plane. In other cases, they can be used to form a polyhedron. What must be true about the angle measures at a common vertex in each case?

iv. Compare tests for symmetries of polygons and other two-dimensional shapes with tests for symmetries of polyhedra and other three-dimensional shapes.

v. Rigidity is often an important consideration in the design of both two-dimensional and three-dimensional shapes. What is the key idea to bracing shapes for rigidity? Why does this work?

Be prepared to share your ideas and reasoning with the class.

Check Your Understanding

Write, in outline form, a summary of the important mathematical concepts and methods developed in this unit. Organize your summary so that it can be used as a quick reference in future units and courses.